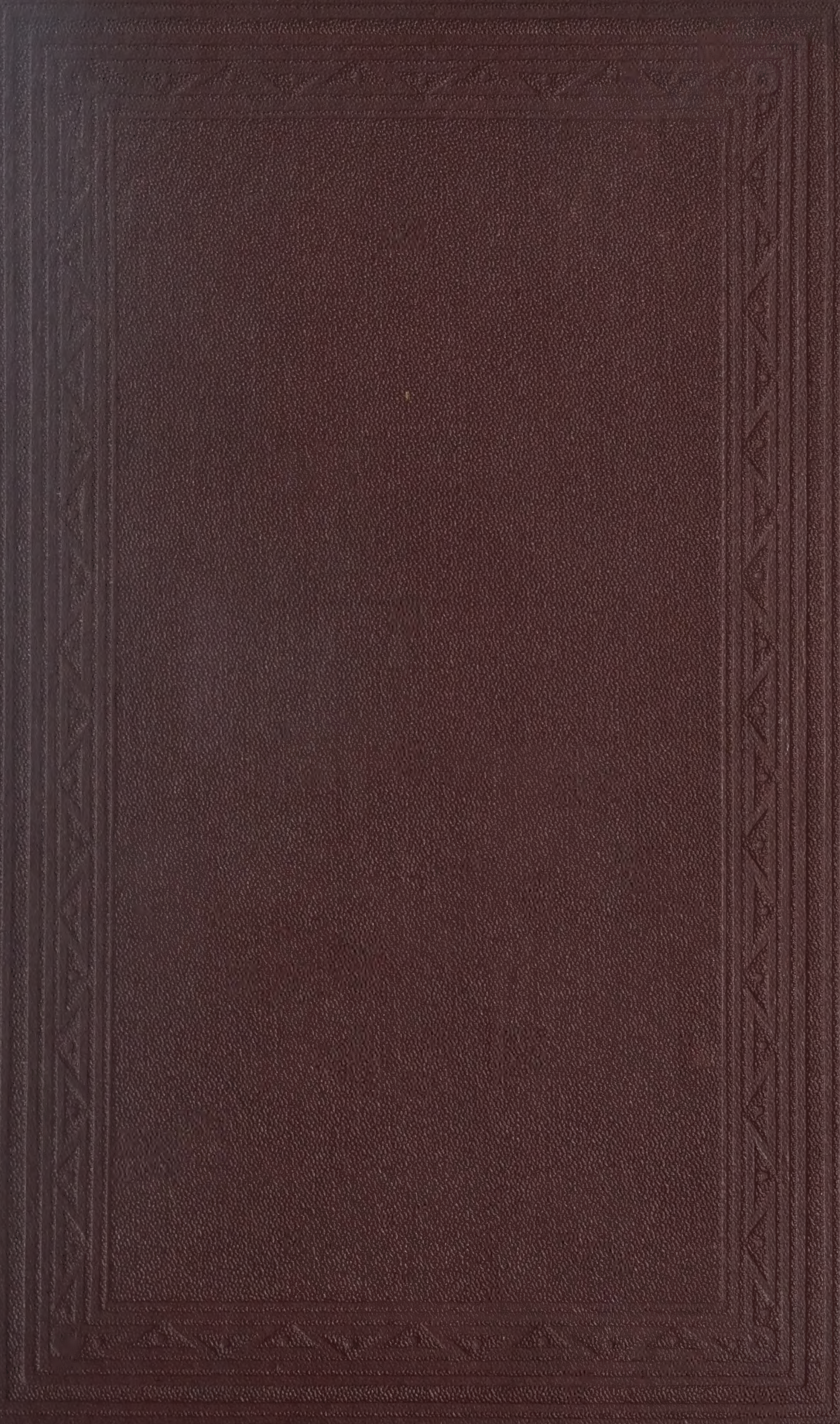

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XI.

FROM NOVEMBER, 1879, TO NOVEMBER, 1880.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XI.

SIXTEENTH SESSION, 1879-80.

November 13*th*, 1879.

Annual General Meeting, held at 22, Albemarle Street.

C. W. MERRIFIELD, Esq., F.R.S., President in the Chair.

The Treasurer (S. Roberts, Esq., F.R.S.) read his Report, which, on the motion of Mr. Terry, seconded by Lieut.-Col. J. R. Campbell, was accepted.

At the request of the Chairman, Mr. R. F. Scott consented to act as Auditor.

The Secretaries' Report was read and adopted.

From the Report of the Secretaries, it appeared that the number of Members during the past Session had increased from 137 to 141, and that the Society had lost by death Prof. W. K. Clifford, F.R.S., Sir John George Shaw Lefevre, F.R.S., and Prof. J. Clerk Maxwell, F.R.S.

The communications made to the Society during the past Session had been as follows:—

"On the Instability of Jets:" Lord Rayleigh, F.R.S.

"On Self-strained Frames of Six joints:" Prof. Crofton, F.R.S.

"On Equivalent Statements, (iii.):" Mr. H. McColl.

"On Forms of Numbers determined by Continued Fractions:" Mr. S. Roberts, F.R.S.

"A Graphic Construction of the Powers of a Linear Substitution:" Prince Camille de Polignac.

"On a Theorem in Elliptic Functions:" Prof. Cayley, F.R.S.

"Coefficients of Induction and Capacity of two Electrified Spheres:" Prof. A. G. Greenhill, M.A.

"On certain Systems of Partial Differential Equations of the First Order with several Dependent Variables:" Mr. H. W. Lloyd Tanner, M.A.

"On the number of Conics which satisfy Five Independent Conditions:" M. Halphen.

"On a Modular Equation, and on the Formula for Four Abelian Functions answering to the Formula for Four Theta Functions:" Prof. H. J. S. Smith, F.R.S.

"Quaternion Proof of Minding's Theorem:" Mr. J. J. Walker, M.A.

"On Differential Equations, Total and Partial; and on a new soluble class of the first and an exceptional case of the second:" Sir J. Cockle, F.R.S.

"Discussion of Two Double Series arising from the Number of Terms in Determinants of certain Forms:" Mr. J. D. H. Dickson, M.A.

"Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general:" the late Prof. Clifford, F.R.S.

"On the Complex whose lines join Conjugate Points of two Correlative Planes:" Dr. Hirst, F.R.S.

"On a Theorem relating to conformable Figures:" Prof. Cayley, F.R.S.

"Method of Constructing, by Pure Analysis, Functions X, Y, &c., which possess the property that $\int XY\,d\sigma = 0$, and such that any given Function can be expanded in the form $\alpha X + \beta Y + \gamma Z + \ldots$:" Mr. Routh, F.R.S.

"The Numerical Calculation of a Class of Determinants, and a Continued Fraction:" Mr. J. D. H. Dickson, M.A.

"Notes on the Reduction of a System of Forces and on Plane Curves:" Mr. J. J. Walker, M.A.

"Notes on Determinants of n Dimensions:" Mr. Lloyd Tanner, M.A.

"On Clifford's Graphs, and on the Twenty-one Coordinates of a Conic in Space:" Dr. Spottiswoode, P.R.S.

"Déduction de différents Théorèmes Géométriques d'un seul Principe Algébrique:" Dr. H. G. Zeuthen.

Besides the above, several minor communications were made to the Society.

Cartes-de-visite likenesses of Mr. C. W. Merrifield, F.R.S., President, and of Mr. S. Roberts, F.R.S., Treasurer, had been presented to the Society's Album.

The same Journals had been subscribed for as in the preceding Session.

The following additions had been made to the number of Journals for which the Society exchanges its Proceedings:—

"The American Journal of Pure and Applied Mathematics;" Baltimore.

"Archives Néerlandaises des Sciences exactes et naturelles, publiées par la Société Hollandaise des Sciences à Harlem;" La Haye.

"Beiblätter zu den Annalen der Physik und Chemie;" Leipzig.

The Meeting then proceeded to the election of the New Council. The Scrutators (Messrs. Lloyd Tanner and McAlister), having examined the balloting lists, declared the following gentlemen duly elected:—

President, C. W. Merrifield, F.R.S.; Vice-Presidents, Prof. Cayley, F.R.S., and Lord Rayleigh, F.R.S.; Treasurer, S. Roberts, F.R.S.; Honorary Secretaries, M. Jenkins, M.A., and R. Tucker, M.A.; other Members, J. W. L. Glaisher, F.R.S., H. Hart, M.A., Dr. Henrici, F.R.S., Dr. Hirst, F.R.S., Dr. Hopkinson, F.R.S., A. B. Kempe, B.A., C. Leudesdorf, M.A., H. W. Lloyd Tanner, M.A., H. M. Taylor, M.A., and J. J. Walker, M.A.

The President briefly alluded to the losses the Society had sustained in the past Session by the deaths of Prof. Clifford, Sir J. G. Shaw Lefevre, and Prof. J. Clerk Maxwell.

A vote of thanks was passed to Mrs. Clifford for the present of four volumes from her late husband's library.

Mr. R. C. Rowe, B.A., was elected a Member of the Society.

The following communications were made to the Society:—

"On the Binomial Equation $x^p-1=0$; Trisection and Quartisection:" Prof. Cayley.

"On Cubic Determinants and other Determinants of Higher Class, and on Determinants of Alternate Numbers:" Mr. R. F. Scott.

"On a Problem of Fibonacci's:" Mr. S. Roberts.

"Notes on a Class of Definite Integrals:" Mr. T. R. Terry.

The following presents were received:—

"Monatsbericht," Juli, August, 1879.

"Beiblätter zu den Annalen der Physik und Chemie," Leipzig, 1879, Band iii., Stück ii.

"Nautical Almanac" for 1883: from Lords Commissioners of Admiralty.

"Cours de Géométrie descriptive de l'Ecole Polytechnique, comprenant les éléments de la Géométrie Cinématique," par A. Mannheim; Paris, 1880: from the Author.

"Washington Observations for 1871." Appendix i.—"Zones of Stars observed at the United States Naval Observatory with the

Meridian Circle in the years 1847, 1848, and 1849;" Washington, 1873.

"Washington Observations for 1869." Appendix ii.—"Zones of Stars observed . . . . with the Mural Circle, in the years 1846—1849," (2 copies); Washington, 1872.

"Washington Observations for 1872." Appendix i.—"Tables of Instrumental Constants and Corrections for the Reduction of Transit Observations made at the United States Naval Observatory;" Washington, 1873.

"Washington Observations for 1870." Appendix iii.—"On the Right Ascension of the Equatorial Fundamental Stars, and the Corrections necessary to reduce the Right Ascensions of different Catalogues to a mean Homogeneous System," by Simon Newcomb; Washington, 1872.

"Washington Observations for 1870." Appendix i.—"Report on the Difference of Longitude between Washington and St. Lewis," by Prof. W. Harkness; Washington, 1872.

"Washington Observations for 1872." Appendix ii.—"Report on the Difference of Longitude between Washington and Detroit, Michigan; Carlin, Nevada; and Austin, Nevada," by J. R. Eastman; Washington, 1879.

"Washington Observations for 1868." Appendix i.—"A Catalogue of 1963 Stars, and of 290 Double Stars observed by the United States Naval Astronomical Expedition to the Southern Hemisphere, during the years 1850—1852."

On the Binomial Equation $x^p-1=0$; *Trisection and Quartisection. By* Prof. Cayley.

[*Read November* 13*th*, 1879.]

The solution of the binomial equation $x^p-1=0$, p a prime number, or, say rather, the equation

$$x^{p-1}+x^{p-2}\dots+x+1=0,$$

depends upon the Jacobian function

$$Fa=x^1+ax^g\dots+a^{p-2}x^{g^{p-2}},$$

where g is a prime root of p, a any root whatever of the equation $u^{p-1}-1=0$. Taking e a factor of $p-1$, and f for the complementary factor (that is, $p-1=ef$), then, if for α we write α^f, or, what is the same thing, taking $\alpha^f, =\beta$, a root of $u^e-1=0$, we have

$$F\beta = X_0+\beta X_1 \dots +\beta^{e-1}X_{e-1},$$

where $X_0, X_1 \dots X_{e-}$ denote each of them a period or sum of $f, =\frac{p-1}{e}$, roots, viz.,

$$\begin{aligned} X_0 &= (1, \quad g^e \quad \dots \; g^{(f-1)e}), \\ X_1 &= (g^e, \quad g^{e+1} \; \dots \; g^{(f-1)e+1}), \\ &\vdots \\ X_{e-1} &= (g^{e-1}, \; g^{2e-1} \; \dots \; g^{fe-1}) \end{aligned}$$

(read $X_0 = x^1+x^{g^e} \dots +x^{g^{(f-1)e}}$, and so for the other functions).

We have, of course, $F(1), = X_0+X_1 \dots +X_{e-1}$, the sum of all the roots $=-1$; and, further, the general property that any rational and integral function of these periods is expressible as a sum

$$a_0X_0+a_1X_1 \dots +a_{-}\, X_{e-}$$

with known coefficients $\quad a_0, a_1 \dots a_{-1}$.

The several cases $e=2, 3, 4, \dots$ may be termed those of the bisection, trisection, quartisection, &c., of the equation; viz.,

$e=2$, there are two periods, X, Y, and $F(-1) = X-Y$;

$e=3$, three periods, X, Y, Z, and $F\gamma = X+\gamma Y+\gamma^2 Z$, if γ is a root of $u^3-1=0$;

$e=4$, four periods, X, Y, Z, W, and $F\delta = X+\delta Y+\delta^2 Z+\delta^3 W$, if δ be a root of $u^4-1=0$.

It is sufficient to attend to the prime roots γ and δ of the equations $u^3-1=0$, $u^4-1=0$, respectively; for, if γ or δ be $=1$, we have simply $F(1), =-1$; and if δ be $=-1$, then the function is $F(-1)$, $= X+Z-(Y+W)$, where $X+Z$ and $Y+W$ are the periods for the bisection. The prime roots δ are of course i and $-i$, and we have

$$\begin{aligned} F(i) \;\; &= X+iY-Z-iW, \\ F(-i) &= X-iY-Z+iW, \end{aligned}$$

respectively.

As regards the bisection, it is known that $(X-Y)^2 = (-)^{\frac{p-1}{4}}p$ ($=+p$ or $-p$, according as p is $\equiv 1$ or 3, mod. 4), and the values of X, Y are thus determined. In what follows, I consider the cases $e=3$ and $e=4$ of the trisection and the quartisection respectively.

It is to be remembered that, not the division into periods, but the order of the periods, depends on the choice of g, a prime root at pleasure of p; and, in what follows, I select the prime root used in Reuschle's

"Tafeln complexer Primzahlen welche aus Wurzeln der Einheit gebildet sind" (4°, Berlin, 1875): viz., these are

$$\begin{aligned} p &= 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\\ &\qquad 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,\\ g &= 2,\ 2,\ 5,\ \ 2,\ \ 2,\ \ 3,\ 2,\ -2,\ \ 2,\ \ 3,\ \ 2,\ \ 6,\ \ 3,\ 10,\ \ 2,\\ &\qquad 2,\ \ 2,\ \ 2,\ 62,\ \ 5,\ \ 3,\ \ 2,\ 30,\ 10, \end{aligned}$$

where I quote the whole series, although I am here only concerned with the values of p which are $\equiv 1$ (mod. 3), or $\equiv 1$ (mod. 4).

The periods are consequently those of Reuschle, viz., X, Y, Z are his η_0, η_1, η_2, and X, Y, Z, W his $\eta_0, \eta_1, \eta_2, \eta_3$: they can of course, without referring to his work, be easily recalculated, but it is, I think, convenient to have for his values of g the series of residues such as are given (for differently selected values of g) in Jacobi's "Canon Arithmeticus" (4°, Berlin, 1839); and I have accordingly taken out of Reuschle, and annex, such a table.

For instance, $p=13$, the powers of g are 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7; and, by writing these down in order in columns of 3 or of 4,

$$\begin{array}{cccc} 1 & 8 & 12 & 5\\ 2 & 3 & 11 & 10\\ 4 & 6 & 9 & 7 \end{array} \qquad\qquad \begin{array}{ccc} 1 & 3 & 9\\ 2 & 6 & 5\\ 4 & 12 & 10\\ 8 & 11 & 7 \end{array}$$

we have the periods X, Y, Z or X, Y, Z, W, belonging to the trisection and the quartisection of $p=13$.

I further remark that the equations which I am concerned with are all given in Reuschle, but in a somewhat different form; thus, $p=13$, quartisection (see p. 13), he has

$$\eta_0^2 = \eta_1 + 2\eta_2, \quad \eta_0\eta_1 = -1-\eta_2, \quad \eta_0\eta_2 = 3+\eta_1+\eta_3, \quad \eta_0\eta_3 = -1-\eta_1,$$

(where observe that here and in every case the value of $\eta_0\eta_3$ is at once obtained from that of $\eta_0\eta_1$ by a mere cyclical interchange of the suffixes, so that the last equation is in fact superfluous); the other equations, using $\eta_0+\eta_1+\eta_2+\eta_3=-1$ to eliminate any constant term which occurs, give my

$$\begin{aligned} X^2 &= (\ \ 0,\ \ \ 1,\ \ \ 2,\ \ \ 0)\,(X,\ Y,\ Z,\ W),\\ XY &= (\ \ 1,\ \ \ 1,\ \ \ 0,\ \ \ 1)\,(\qquad ,, \qquad),\\ XZ &= (-3,\ -2,\ -3,\ -2)\,(\qquad ,, \qquad). \end{aligned}$$

Similarly, in the case of a trisection, the equation for $\eta_0\eta_2$ is superfluous, and the other equations give my values of X^2 and XY.

Reuschle gives also, and I take from him, the cubic and quartic equations (such as $p=13$, $\eta^3+\eta^2-4\eta+1=0$, $\eta^4+\eta^3+2\eta^2-4\eta+3=0$), which determine the periods in the trisections and quartisections respectively.

Many of the results obtained accord with, and furnish exemplifications of general theorems contained in Jacobi's memoir, "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie,"—Crelle, t. xxx. (1846), pp. 166–189.

Trisection, $e=3$; $p\equiv 1$ (*mod.* 3).

We have three periods X, Y, Z; and we thence obtain

$$\begin{aligned} X^2 &= (a,\ b,\ c)(X,\ Y,\ Z), \\ XY &= (f,\ g,\ h)(\quad ,, \quad), \end{aligned}$$

the coefficients a, b, c, f, g, h being determinate integers. And, by cyclical interchanges, we obtain equations which may be written

$$\begin{aligned} X^2 &= a,\ b,\ c, \\ Y^2 &= c,\ a,\ b, \\ Z^2 &= b,\ c,\ a, \\ XY &= f,\ g,\ h, \\ YZ &= h,\ f,\ g, \\ ZX &= g,\ h,\ f; \end{aligned}$$

viz., here and elsewhere the coefficients a, b, c are written to denote the sum $aX+bY+cZ$. It is easy to see that

$$f+g+h=\tfrac{1}{3}(p-1);$$

in fact, a period contains $\frac{1}{3}(p-1)$ terms, and in two consecutive periods X, Y, there are no terms the product of which is unity; hence XY contains $\frac{1}{9}(p-1)^2$ terms, each a power of x, and the sum $XY+YZ+ZX$ contains $\frac{1}{3}(p-1)^2$ such terms, being in fact the sum $X+Y+Z$ taken $\frac{1}{3}(p-1)$ times; whence the relation in question.

Hence also $\qquad YZ+ZX+XY=-\frac{1}{3}(p-1)$.

From the equation $X+Y+Z=-1$, multiplying by X, and for X^2, XY, XZ substituting their values, we obtain an expression

$$(a+f+g+1)\,X+(b+g+h)\,Y+(c+h+f)\,Z,$$

which must identically vanish; viz., the three coefficients must be each of them $=0$; or we must have

$$\begin{aligned} a &= -f-g-1, \\ b &= -g-h, \\ c &= -h-f; \end{aligned}$$

so that, taking f, g, h as known, the other coefficients a, b, c are given in terms of these. The equations give

$$a+b+c=-2(f+g+h)-1.$$

We have $X.YZ = Y.ZX$; that is, $X(h, f, g) = Y(g, h, f)$; or, substituting for X^2, XY, &c. their values,

$$\begin{aligned} h(a,\ b,\ c) &= \ \ g(f,\ g,\ h) \\ +f(f,\ g,\ h) &\quad +h(c,\ a,\ b) \\ +g(g,\ h,\ f) &\quad +f(h,\ f,\ g); \end{aligned}$$

that is,

$$\begin{aligned} ah+f^2+g^2 &= gf+ch+fh, \\ bh+fg+gh &= g^2+ah+f^2, \\ ch+fh+fg &= gh+bh+fg, \end{aligned}$$

equations which reduce themselves to the single equation

$$gh+hf+fg+h = f^2+g^2+h^2;$$

and this is the only relation obtainable by consideration of the three equal values $X.YZ$, $Y.ZX$, $Z.XY$. Moreover, this equation being satisfied, the six functions in the three equations become each of them $=fg-h^2$; or we have

$$XYZ = (fg-h^2,\ fg-h^2,\ fg-h^2);$$

that is,

$$XYZ = h^2-fg.$$

We have

$$\begin{aligned} F\gamma.F\gamma^2 &= \ \ X^2+Y^2+Z^2-YZ-ZX-XY \\ &= \ \ (a+b+c-f-g-h)(X+Y+Z) \\ &= -(a+b+c)+(f+g+h) \\ &= \ \ 3(f+g+h)+1; \end{aligned}$$

that is,

$$F\gamma.F\gamma^2 = p.$$

We have, moreover,

$$\begin{aligned} (F\gamma)^2 = &\quad X^2+2YZ+\gamma(Z^2+2XY)+\gamma^2(Y^2+2ZX) \\ = &\quad [(a, b, c)+2(h, f, g)] \\ &+\gamma\,[(b, c, a)+2(f, g, h)] \\ &+\gamma^2[(c, a, b)+2(g, h, f)], \end{aligned}$$

which is $=\{(a+2h)+\gamma(b+2f)+\gamma^2(c+2g)\}(X+\gamma^2Y+\gamma Z)$,

as is at once seen by comparing the coefficients of X, Y, Z respectively.

Hence, writing

$$\begin{aligned} &a+2h+\gamma(b+2f)+\gamma^2(c+2g) \\ &= a+2h+\gamma(b+2f)-(1+\gamma)(c+2g) \\ &= A+B\gamma, \end{aligned}$$

we have

$$\begin{aligned} A &= a+2h-c-2g = 3h-3g-1 \\ B &= b+2f-c-2g = 3f-3g. \end{aligned}$$

We have

$$(F\gamma)^2 = (A+B\gamma)\,F(\gamma^2),$$

and thence, writing γ^2 for γ,

$$(F\gamma^2)^2 = (A+B\gamma^2)\,F\gamma,$$

equations which give

$$F\gamma\,.\,F\gamma^2, \;=p,\; = (A+B\gamma)\,(A+B\gamma^2)\,;$$

or, say $p = A^2-AB+B^2$; viz., p has the complex factor

$$A+B\gamma, \;= 3h-3g-1+\gamma\,(3f-3g).$$

Hence also
$$(F\gamma\,)^3 = p\,(A+B\gamma\,),$$
$$(F\gamma^2)^3 = p\,(A+B\gamma^2),$$

and, as before,
$$F\gamma\,.\,F\gamma^2 = p\,;$$

which equations determine $F\gamma$, $F\gamma^2$, and from these and $F(1) = -1$ we obtain the periods X, Y, Z; we have thus, in fact, the solution of the cubic equation which gives these periods. We have already found the coefficients of this cubic equation, viz.,

$$X+Y+Z=-1, \quad YZ+ZX+XY = -\tfrac{1}{3}(p-1), \quad XYZ = h^2-fg\,;$$

the equation thus is

$$\eta^3+\eta^2-\tfrac{1}{3}(p-1)\,\eta+(fg-h^2)=0.$$

As already remarked, the values of a, b, c; f, g, h, and the equations in η, are in effect given in Reuschle; the complex factors of p, as given p. 1 ($7 = 2\gamma-3\gamma^2$, &c.), when reduced to the form $A+B\gamma$, are not identical with the $A+B\gamma$ of the foregoing theory; viz., this $A+B\gamma$ is not Reuschle's selected primary form. I give, in the annexed table for the primes 7, 13 ... to 97, the values from Reuschle of a, b, c; f, g, h, and of the coefficients of the η-equation; also the values of A and B derived from f, g, h by the foregoing formulæ. It will be seen that all the values are consistent with the theory.

TABLE FOR THE TRISECTION.

p	a,	b,	c	$\eta^3+\eta^2+$		A	B	*Page in Reuschle.*
	f,	g,	h	η^1	η^0			
7	−2	−1	−2	−2	−1	2	3	*p.* 6
	1	0	1					
13	−4	−3	−2	−4	−1	−4	−3	*p.* 15
	1	2	1					
19	−4	−5	−4	−6	−7	2	−3	*p.* 26
	1	2	3					
31	−7	−6	−8	−10	−8	5	6	*p.* 45
	4	2	4					
37	−8	−10	−7	−12	11	−4	3	*p.* 54
	5	4	3					
43	−11	−8	−10	−14	8	−1	6	*p.* 69
	6	4	4					
61	−14	−13	−15	−20	−9	−4	−9	*p.* 97
	5	8	7					
67	−16	−13	−16	−22	5	2	9	*p.* 105
	9	6	7					
73	−16	−18	−15	−24	−27	−1	−9	*p.* 128
	6	9	9					
79	−20	−17	−16	−26	41	−10	−3	*p.* 138
	9	10	7					
97	−20	−23	−22	−32	−79	11	3	*p.* 168
	10	9	13					

Quartisection, $e=4$, $p\equiv 1$ (*mod.* 4).

We have four periods X, Y, Z, W, and we obtain

$$\begin{aligned} X^2 &= (a, b, c, d)(X, Y, Z, W), \\ XY &= (f, g, h, k)(\quad,, \quad), \\ XZ &= (l, m, l, m)(\quad,, \quad), \end{aligned}$$

the coefficients being determinate integers. It can be shown that $l+m=\frac{1}{8}(p-1)$ or $-\frac{1}{8}(3p+1)$ according as $p\equiv 1$ or 5 (mod. 8). And then, by cyclical interchanges,

$$\begin{aligned} X^2 &= a,\ b,\ c,\ d, \\ Y^2 &= d,\ a,\ b,\ c, \\ Z^2 &= c,\ d,\ a,\ b, \\ W^2 &= b,\ c,\ d,\ a, \\ XY &= f,\ g,\ h,\ k, \\ YZ &= k,\ f,\ g,\ h, \\ ZX &= h,\ k,\ f,\ g, \\ XW &= g,\ h,\ k,\ f, \\ XZ &= l,\ m,\ l,\ m, \\ YW &= m,\ l,\ m,\ l. \end{aligned}$$

We have, in like manner as for the trisection,

$$f+g+h+k=\tfrac{1}{4}(p-1),$$

and so also the expression for

$$\Sigma XY, = XY+XZ+XW+YZ+YW+ZW$$

is $$= -(f+g+h+k+l+m) = -\tfrac{1}{4}(p-1)-l-m;$$

and, in virtue of the foregoing value of $l+m$, this is $=-\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p\equiv 1$ or 5 (mod. 8).

Again, from the equation $X+Y+Z+W=-1$, multiplying by X and reducing,

$$\begin{aligned} a &= -1-f-g-l, \\ b &= \quad\ -g-h-m, \\ c &= \quad\ -h-k-l, \\ d &= \quad\ -k-f-m, \end{aligned}$$

and thence $a+b+c+d=-1-2(f+g+h+k)-2(l+m)$,

and $$a-b+c-d=-1+2(m-l).$$

We have $X\,.\,YZ = Y\,.\,ZX = Z\,.\,XY$, that is,

$$X\,(k, f, g, h) = Y\,(l, m, l, m) = Z\,(f, g, h, k),$$

and thence

$$\begin{array}{lll}
k\,(a,\ b,\ c,\ d\,) = & l\ \ (f,\ g,\ h,\ k) = & f\ (l,\ m,\ l,\ m) \\
+f\ (f, g,\ h, k\,) & +m\,(d,\ a,\ b,\ c) & +g\,(k, f,\ g, h\,) \\
+g\ (l,\ m, l,\ m) & +l\ \ (k,\ f,\ g,\ h) & +h\,(c,\ d,\ a,\ b\,) \\
+h\,(g,\ h,\ k, f\ ) & +m\,(m, l,\ m,\ l) & +k\,(h,\ k,\ f,\ g\,),
\end{array}$$

that is,
$$\begin{aligned}
ka+f^2+gl+gh &= lf+md+lk+m^2 = lf+gk+ch+kh, \\
kb+fg+gm+h^2 &= lg+am+lf+ml = fm+fg+hd+k^2, \\
kc+fh+gl+hk &= lh+mb+lg+m^2 = fl+g^2+ah+kf, \\
kd+fk+gm+fh &= kl+mc+lh+lm = fm+gh+bh+gk,
\end{aligned}$$

in which equations a, b, c, d may be regarded as having their foregoing values.

One of these equations is

$$kc+fh+gl+hk = lf+g^2+ah+kf,$$

that is, $-k\,(h+k+l)+fh+gl+hk = lf+g^2-h\,(f+g+l+1)+kf,$

or, reducing, $\quad l\,(g+h-f-k) = g^2+k^2-2hf-hg-h+kf,$

which gives l.

Again, another equation is

$$kb+fg+gm+h^2 = fm+fg+hd+k^2,$$

that is, $-k\,(g+h+m)+fg+gm+h^2 = fm+fg-h\,(k+f+m)+k^2,$

or, reducing, $\quad m\,(g+h-f-k) = k^2-h^2+gk-hf,$

which gives m.

And we have also

$$md+lk+m^2 = gk+ch+kh,$$

that is, $\quad -m\,(k+f+m)+lk+m^2 = gk+kh-h\,(h+k+l),$

or, reducing, $\quad l\,(k+h)-m\,(f+k) = gk-h^2.$

Substituting herein for l, m their values, we have

$$\begin{gathered}
(k+h)\,[g^2+k^2-2hf-hg+kf-h]-(f+k)\,[k^2-h^2+gk-hf] \\
+(h^2-gk)\,[g+h-f-k] = 0.
\end{gathered}$$

In this equation the only terms of the second order are $-h\,(h+k)$, which contain the factor h; the terms of the third order contain this same factor h, and throwing it out, and reducing, the equation is found to be

$$(g-k)^2+(h-f)^2 = h+k,$$

or, as it may also be written,

$$g^2+k^2-2hf-h+(h^2+f^2-2gk-k)=0\,;$$

and the foregoing values of l, m are

$$l = \frac{(g^2+k^2-2hf-h)-(gh-kf)}{g+h-k-f},$$

$$m = \frac{k^2-h^2+gk-hf}{g+h-k-f}\,;$$

and by means of these three equations all the foregoing equations are satisfied.

We have $F(i)\,F(i^3)=(X-Z)^2+(Y-W)^2$

$$\begin{aligned} &= X^2+Y^2+Z^2+W^2-2(XZ+YW) \\ &= -(a+b+c+d)+2(l+m)\,; \end{aligned}$$

or, substituting for a, b, c, d, this is

$$= 1+2(f+g+h+k)+4(l+m),$$

viz., it is $\quad = \tfrac{1}{2}(p+1)+4(l+m)\,;$

or, substituting for $l+m$ its before-mentioned value, then, according as $p\equiv 1$ or 5 (mod. 8), the value is $=p$ or $-p$; that is, we have

$$F(i)\,F(i^3)=(-)^{\frac{p-1}{4}}\,p.$$

Again, we have

$$\begin{aligned} (Fi)^2 &= (X+iY-Z-iW)^2 \\ &= X^2-Y^2+Z^2-W^2-2XZ+2YW \\ &\qquad +2i\,(XY-YZ+ZW-WX) \\ &= \;\; a-b+c-d+2(m-l)+2(f-g+h-k)i\}\,(X-Y+Z-W) \\ &= (A+Bi)\,F(-1), \end{aligned}$$

where

$$\begin{aligned} A &= a-b+c-d+2(m-l), = -1+4(m-l), \\ B &= 2(f-g+h-k)\,; \end{aligned}$$

or, since $X-Y+Z-W=F(-1)$, this equation is

$$(Fi)^2=(A+Bi)\,F(-1),$$

and similarly $\quad (Fi^3)^2=(A-Bi)\,F(-1).$

Moreover, $\quad [F(-1)]^2=(-)^{\frac{p-1}{2}}\,p, \;\; =p\,;$

and we have therefore

$$(\pm p)^2=(A^2+B^2)\,p, \text{ that is, } A^2+B^2=p\,;$$

or the expression $A+Bi$ determined as above is a complex factor of p.

We may investigate the quartic equation for the determination of the periods X, Y, Z, W. The values of $X+Y+Z+W$ and $XY+XZ+XW+YZ+YW+ZW$ are already known: for the next coefficient $XYZ+XYW+XZW+YZW$, we have $XYZ=(\alpha, \beta, \gamma, \delta)$, where each of the coefficients $\alpha, \beta, \gamma, \delta$ is given under three different forms: the values of YZW, ZWX, WXY are $(\delta, \alpha, \beta, \gamma)$, $(\gamma, \delta, \alpha, \beta)$, $(\beta, \gamma, \delta, \alpha)$; and the required sum therefore is

$$(\alpha+\beta+\gamma+\delta)(X+Y+Z+W), =-(\alpha+\beta+\gamma+\delta).$$

Taking the first expressions of these coefficients respectively, we have

$$\begin{aligned}\alpha+\beta+\gamma+\delta = \; & k(a+b+c+d)\\ & +f(f+g+h+k)\\ & +2g(l+m)\\ & +h(f+g+h+k),\end{aligned}$$

$$=k\{-1-\tfrac{1}{2}(p-1)-2(l+m)\}+(f+h)\{\tfrac{1}{4}(p-1)\}+2g(l+m),$$
$$=2(g-k)(l+m)+\tfrac{1}{4}(f+h)(p-1)-\tfrac{1}{2}k(p+1).$$

We find $XYZW$ most readily as the product of XZ and YW; we thus obtain

$$\begin{aligned}XYZW = \; & lm(X^2+Y^2+Z^2+W^2+2XZ+2YW)\\ & \quad +(l^2+m^2)(XY+XW+YZ+ZW),\\ = \; & lm(-a-b-c-d-2l-2m)-(l^2+m^2)(f+g+h+k),\\ = \; & lm\{1+2(f+g+h+k)\}-(l^2+m^2)(f+g+h+k);\end{aligned}$$

or, substituting for $f+g+h+k$ its value $\frac{1}{4}(p-1)$, this is

$$lm-\tfrac{1}{4}(l-m)^2(p-1), =\tfrac{1}{4}\{(l+m)^2-(l-m)^2p\}.$$

Hence the required equation, roots X, Y, Z, W, is

$$\begin{aligned}& \eta^4+\eta^3\\ & -\eta^2\{\tfrac{1}{4}(p-1)+l+m\}\\ & +\eta\{\tfrac{1}{4}(f+h)(p-1)-\tfrac{1}{2}k(p+1)+2(g-k)(l+m)\}\\ & +lm-\tfrac{1}{4}(l-m)^2(p-1)\\ & =0,\end{aligned}$$

where, for the sake of having a single formula, I have retained $l+m$ in place of its value $=-\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p\equiv 1$ or 5 (mod. 8).

We thus have the following:—

TABLE FOR THE QUARTISECTION.

p	a f l	b g m	c h	d k	$\eta^4+\eta^3+$ η^2	η^1	η^0	A	B	*Page in Reuschle.*
5	0	1	0	0	1	1	1	−1	−2	*p. 2*
	0	0	0	1						
	−1	−1								
13	0	1	2	0	2	−4	3	3	−2	*p. 13*
	1	1	0	1						
	−3	−2								
17	−4	−2	−3	−4	−6	−1	1	−1	4	*p. 19*
	2	0	1	1						
	1	1								
29	2	3	0	2	4	20	23	−5	−2	*p. 36*
	1	1	2	3						
	−5	−6								
37	2	1	2	4	5	7	49	−1	6	*p. 53*
	2	2	4	1						
	−7	−7								
41	−10	−6	−7	−8	−15	18	4	−5	4	*p. 61*
	4	2	2	2						
	3	2								
53	2	3	6	2	7	−43	47	7	−2	*p. 80*
	4	4	2	3						
	−11	−9								
61	4	3	2	6	8	42	117	−5	6	*p. 96*
	3	3	6	3						
	−11	−12								
73	−16	−13	−12	−14	−27	−41	2	3	8	*p 126*
	6	5	5	2						
	4	5								
89	−19	−18	−16	−14	−33	39	8	−5	−8	*p. 152*
	4	8	5	5						
	6	5								
97	−22	−16	−17	−18	−36	91	−61	−9	4	*p 167*
	8	6	5	5						
	7	5								

TABLE OF THE POWERS OF REUSCHLE'S SELECTED PRIME ROOTS.

	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	2	2	5	2	2	3	2	21	2	3	2	6	3	10	2	2	2	2	62	5	3	2	30	10	1
2		4	4	4	4	9	4	4	4	9	4	36	9	6	4	4	4	4	10	25	9	4	10	3	2
3	...	3	6	8	8	10	8	15	8	27	8	11	27	13	8	8	8	8	52	52	27	8	33	30	3
4			2	5	3	13	16	16	16	19	16	25	38	36	16	16	16	16	29	41	2	16	11	9	4
5	...	...	3	10	6	5	13	14	3	26	32	27	28	31	32	32	32	32	23	59	6	32	63	90	5
6				9	12	15	7	18	6	16	27	39	41	28	11	5	3	64	6	3	18	64	21	27	6
7				7	11	11	14	10	12	17	17	29	37	45	22	10	6	61	17	15	54	45	7	76	7
8				3	9	16	9	3	24	20	34	10	25	27	44	20	12	55	60	2	4	7	32	81	8
9	...	...	...	6	5	14	18	17	19	29	31	19	32	35	35	40	24	43	28	10	12	14	70	34	9
10					10	8	17	12	9	25	25	32	10	21	17	21	48	19	32	50	36	28	53	49	10
11	...	...	...	...	7	7	15	22	18	13	13	28	30	22	34	42	35	38	67	31	29	56	77	5	11
12						4	11	2	7	8	26	4	4	32	15	25	9	9	36	9	8	29	85	50	12
13						12	3	19	14	24	15	24	12	38	30	50	18	18	31	45	24	58	58	15	13
14						2	6	8	28	10	30	21	36	4	7	41	36	36	5	6	72	33	49	53	14
15	...	...	...	...	...	6	12	7	27	30	23	3	22	40	14	23	11	5	26	30	58	66	46	45	15
16							5	9	25	28	9	18	23	24	28	46	22	10	50	4	16	49	45	62	16
17	...	...	...	...	...	...	10	5	21	22	18	26	26	5	3	33	44	20	47	20	48	15	15	38	17
18								13	13	4	36	33	35	3	6	7	27	40	3	27	65	30	5	89	18
19								20	26	12	35	34	19	30	12	14	54	13	44	62	37	60	61	17	19
20								6	23	5	33	40	14	18	24	28	47	26	30	18	32	37	50	73	20
21	...	...	...	...	...	...	...	11	17	15	29	35	42	39	48	56	33	52	14	17	17	74	76	51	21
22									5	14	21	5	40	14	43	53	5	37	16	12	51	65	55	25	22
23									10	11	5	30	34	46	33	47	10	7	69	60	74	47	48	56	23
24									20	2	10	16	16	37	13	35	20	14	18	8	64	11	16	75	24
25									11	6	20	14	5	41	26	11	40	28	51	40	34	22	35	71	25
26									22	18	3	2	15	34	52	22	19	56	38	54	23	44	71	31	26
27	...	...	...	...	...	...	...	...	15	23	6	12	2	11	51	44	38	45	13	51	69	5	83	19	27
28										7	12	31	6	16	49	29	15	23	25	36	48	10	87	93	28
29	...	...	...	...	...	...	...	...	...	21	24	22	18	19	45	58	30	46	59	34	68	20	29	57	29
30											11	9	11	2	37	57	60	25	37	24	46	40	69	85	30
31											22	13	33	20	21	55	59	50	22	47	59	80	23	74	31
32											7	37	13	12	42	51	57	33	15	16	19	77	67	61	32
33											14	17	39	26	31	43	53	66	7	7	57	71	52	28	33
34											28	20	31	25	9	27	45	65	8	35	13	59	47	86	34
35	...	...	...	...	...	...	...	...	...	...	19	38	7	15	18	54	29	63	70	29	39	35	75	84	35
36												23	21	9	36	49	58	59	9	72	38	70	25	64	36
37												15	20	43	19	39	55	51	61	68	35	57	38	58	37
38												8	17	7	38	19	49	35	19	48	26	31	72	95	38
39	...	...	...	...	...	...	...	...	...	...	...	7	8	23	23	38	37	3	42	21	78	62	24	77	39
40													24	42	46	17	13	6	48	32	76	41	8	91	40
41	...	...	...	...	...	...	...	...	...	...	...	...	29	44	39	34	26	12	65	14	70	82	62	37	41
42														11	25	9	52	24	45	70	52	81	80	79	42
43														29	50	18	43	28	11	58	77	79	86	14	43
44														8	47	36	25	29	43	71	73	75	88	43	44
45	...	...	...	...	...	...	...	...	...	...	...	...	...	33	41	13	50	58	39	63	61	67	59	42	45
46															29	26	39	49	4	23	25	51	79	32	46
47															5	52	17	31	35	42	75	19	56	29	47
48															10	45	34	62	40	64	67	38	78	96	48
49															20	31	7	57	66	28	43	76	26	87	49
50															40	3	14	47	45	67	50	69	68	94	50
51	...	...	...	...	...	...	...	...	...	...	...	...	...	...	27	6	28	27	21	43	71	55	82	67	51
52																12	56	54	24	69	55	27	57	88	52
53																24	51	41	68	53	7	54	19	7	53
54																48	41	15	27	46	21	25	36	70	54

	59	61	67	71	73	79	83	89	97	
55	37	21	30	41	11	63	50	12	21	55
56	15	42	60	57	55	31	17	4	16	56
57	30	23	53	55	56	14	34	31	63	57
58		46	39	2	61	42	68	40	48	58
59		31	11	53	13	47	53	43	92	59
60			22	20	65	62	23	44	47	60
61			44	33	33	28	46	74	82	61
62			21	58	19	5	9	84	44	62
63			42	46	22	15	18	28	52	63
64			17	12	37	45	36	39	35	64
65			34	34	39	56	72	13	59	65
66				49	49	10	61	34	8	66
67				56	26	30	39	41	80	67
68				64	57	11	78	73	24	68
69				63	66	33	73	54	46	69
70					38	20	63	18	72	70
71					44	60	43	6	41	71
72						22	3	2	22	72
73						26	6	60	26	73
74						40	12	20	66	74
75						41	24	66	78	75
76						44	48	22	4	76
77						53	13	37	40	77
78							26	42	12	78
79							52	14	23	79
80							21	64	36	80
81							42	51	69	81
82								17	11	82
83								65	13	83
84								81	33	84
85								27	39	85
86								9	2	86
87								3	20	87
88									6	88
89									60	89
90									18	90
91									83	91
92									54	92
93									55	93
94									65	94
95									68	95

On Cubic Determinants and other Determinants of Higher Class, and on Determinants of Alternate Numbers. By R. F. Scott, M.A.

[*Read Nov.* 13*th*, 1879.]

1. Determinants with three or more suffixes have as yet received little attention in England. So far as I know, the only recent memoir relating to them is one by Mr. H. W. Lloyd Tanner, "Proceedings of London Mathematical Society," x., p. 167 (1879). The functions

were originally considered by Prof. Cayley as sums of determinants, in his memoir "On the Theory of Determinants," "Cambridge Philosophical Transactions," viii., p. 75 (1849).

On the Continent the literature of the subject is as follows:—

A. Armenante, "Sui determinanti cubici," "Giornale di Mathematiche," vi., p. 175 (1866).

E. Padova, "Sui determinanti cubici, *ibid.*" (1868).

A. de Gasparis, "Sopra due teoremi dei determinanti a tre indici ed un altra maniera di formazione de gli elementi di un determinante ad m indici," "Rendiconti dell' Accad. delle Scienze fis. e mat. di Napoli," vii., p. 118 (1868).

In the last two memoirs it is stated that a professor at the University of Naples, concealing his name under the anagram *Jean Blaise Grandpas*, laid the foundation of a theory of determinants with three suffixes, in a pamphlet entitled, "Sur les déterminants dont les éléments ont plusieurs indices" (1861).*

G. Zehfuss, "Ueber eine Erweiterung des Begriffes der Determinanten." Frankfurt a. M. (1868).

G. Garbieri, "Determinanti formati di elementi con un numero qualunque d'indici," "Giornale," xv., p. 89 (1875).

There is also a memoir by *Dahlander*, "Om en klass funktioner hvilka ega flera egenskaper analoga med determinanternes" (Öfvers. af K. Vet. Akad. Förh., 1863, p. 300), which I have not seen.

I propose in the present memoir to show how the representation of an ordinary determinant by means of a product of Grassmann's "Alternate Numbers,"† can be extended to cubic determinants and other determinants of higher class.

2. If we have n^3 elements with three suffixes, such as

$$a_{ijk} \quad (i, j, k = 1, 2 \dots n)$$

we may arrange them in the form of a cube at the intersections of three sets of rectangular planes. Taking, then, the principal diagonal of elements

$$a_{111}\, a_{222} \dots a_{nnn} \quad \dots\dots\dots\dots\dots\dots (1),$$

we can form from it a cubic determinant as follows:—

In the term (1) permutate the second and third suffixes in all possible ways, and give to each term the sign corresponding to the sum of the number of inversions in the series of second and third suffixes. The sum of all the $\{n\,!\}^2$ terms thus formed is called a cubic determinant.

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* It has been pointed out to me that this was probably de Gasparis himself (Annibale de Gasparis).

† I notice that Mr. Spottiswoode ("Proceedings of London Mathematical Society," vii., p. 100) refers to this representation as "Hankel's Fundamental Theorem." The theorem is, however, due to Grassmann, and is given by him, "Die Ausdehnungslehre," p. 37 (1862). I have not seen his earlier work of 1844.

I shall suppose that the series of elements which have the same first suffix stand in the same horizontal plane, and, for brevity, call these planes *strata*.

Let

$$\begin{gathered} e_1,\ e_2\ \ldots\ e_n, \\ \epsilon_1,\ \epsilon_2\ \ldots\ \epsilon_n, \end{gathered}$$

be two independent sets of alternate units of the n^{th} order. Then, with the elements in the i^{th} stratum of the cubic determinant, we can form the lineo-linear function of these numbers:

$$\begin{array}{l} \,a_{i11}\,e_1\,\epsilon_1 + a_{i12}\,e_1\,\epsilon_2 + \ldots + a_{i1n}\,e_1\,\epsilon_n, \\ + a_{i21}\,e_2\,\epsilon_1 + a_{i22}\,e_2\,\epsilon_2 + \ldots + a_{i2n}\,e_2\,\epsilon_n, \\ + \quad \ldots \quad\quad \ldots \quad\quad \ldots \quad\quad \ldots \\ + a_{in1}\,e_n\,\epsilon_1 + a_{in2}\,e_n\,\epsilon_2 + \ldots + a_{inn}\,e_n\,\epsilon_n. \end{array}$$

There will be n such functions corresponding to the n strata of the determinant; the product of these n functions is the cubic determinant. This is evident as soon as stated, for, owing to the properties of alternate numbers, no term can occur having two second or two third suffixes alike, and when the products of the units are brought to the orders

$$e_1\,e_2\ \ldots\ e_n,\quad \epsilon_1\,\epsilon_2\ \ldots\ \epsilon_n,$$

each term will have its proper sign. As usual, the product of all the units of a set in a definite order, is taken equal to unity.

3. We are now in a position to represent a determinant of elements with any number of suffixes by a like product. I shall call a determinant of elements with p suffixes, one of the p^{th} class. Thus ordinary determinants are determinants of the second class, a determinant of the first class consists of a single element.

A determinant of the p^{th} class and n^{th} order consists of n^p elements, which it is convenient to consider as arranged in a rectangular parallelosсhemon of p dimensions, at the intersections of p systems of mutually perpendicular planes. Its representation as a product of alternate numbers is obtained by forming, with all the elements having the same first suffix i (these lie in the same rectangular parallelosсhemon of $p-1$ dimensions), a function linear in $p-1$ independent sets of alternate units of order n.

For the determinant of the fourth class, for example, using the umbral notation, the sum for the elements standing in the i^{th} cube is

$$\Sigma\ (ijkl)\ e_j\,\epsilon_k\,\eta_l,$$

where j, k, l take all the values $1, 2\ \ldots\ n$. The determinant is equal to the product of these sums for the values $1, 2\ \ldots\ n$ of i.

4. We can readily evaluate some special forms of cubic determinants, and others of higher class, by means of this representation. For example, if in a cubic determinant all the elements of the i^{th} stratum are equal to a_i, except that in the principal diagonal of the cube, which is equal to x_i, so that $a_{ijk} = a_i, \quad a_{iii} = x_i,$

then the value of the determinant is given by

$$D = \Pi\left[a_i EE' + (x_i - a_i)\, e_i \epsilon_i\right],$$

where

$$E = e_1 + e_2 + \ldots + e_n,$$
$$E' = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_n.$$

Since E, E' are alternate numbers, all their powers vanish; and if

$$u = (x_1 - a_1)(x_2 - a_2) \ldots (x_n - a_n),$$
$$D = u + a_1 \frac{du}{dx_1} + a_2 \frac{du}{dx_2} + \ldots + a_n \frac{du}{dx_n}.$$

As another example of the same kind, let all the elements in the i^{th} stratum be equal to a_i, except those which lie in the principal diagonal of the stratum itself, these n elements being equal to x_i. Then the value of the determinant is $D = \Pi\left[(x_i - a_i)\, F + a_i EE'\right],$

where E, E' have their previous meanings, and

$$F = e_1 \epsilon_1 + e_2 \epsilon_2 + \ldots + e_n \epsilon_n.$$

Now, binary products of alternate units, such as the terms of F, are "concurrent numbers," and clearly

$$F^n = n!, \quad F^{n-1} EE' = n!.$$

Hence

$$D = n!\left\{u + a_1 \frac{du}{dx_1} + a_2 \frac{du}{dx_2} + \ldots + a_n \frac{du}{dx_n}\right\}.$$

It is clear that the first of these two theorems holds for determinants of all classes, and is, in fact, the generalisation of Sardi's theorem for ordinary determinants; viz.,

$$\begin{vmatrix} x_1, & a_2, & a_3 & \ldots \\ a_1, & x_2, & a_3 & \ldots \\ a_1, & a_2, & x_3 & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} = u + a_1 \frac{du}{dx_1} + \ldots + a_n \frac{du}{dx_n},$$

("Un teorema su' Determinanti," *Giornale*, vi., p. 357).

The second theorem, however, holds only for determinants whose class is odd, if we except determinants of the second class, which are, in some respects, *sui generis*. For example, for the determinant of the fourth class, in which all the elements in the i^{th} cube are a_i, except those in its principal diagonal, which are x_i,

$$D = \Pi\,\{(x_i - a_i)\,F + a_i E_1 E_2 E_3\},$$

$$E_1 = e_1 + \ldots + e_n,\;\; E_2 = \epsilon_1 + \ldots + \epsilon_n,\;\; E_3 = \eta_1 + \ldots + \eta_n,$$

$$F = e_1 \epsilon_1 \eta_1 + e_2 \epsilon_2 \eta_2 + \ldots + e_n \epsilon_n \eta_n.$$

Ternary products of alternate units, such as the terms of F, are alternate numbers; hence all powers of F vanish, as do also those of E_1, E_2, E_3; hence the determinant vanishes.

The following additional examples are of a different kind. The value of the cubic determinant D, whose elements are given by

$$a_{ijk} = a_i + a_j + a_k, \quad a_{iii} = 0,$$

is

$$\frac{(-1)^n D}{3^n a_1 a_2 \ldots a_n} = 1 - n - \tfrac{2}{9}\Sigma\,\frac{(a_i - a_k)^2}{a_i a_k}.$$

That whose elements are given by

$$a_{ijk} = \cos(a_i + a_j + a_k), \quad a_{iii} = 0,$$

is

$$\frac{(-1)^{n-1} D}{\cos 3a_1 \cos 3a_2 \ldots \cos 3a_n} = n - 1 - 2\Sigma\,\frac{\cos(a_i + a_k)\sin^2(a_i - a_k)}{\cos 3a_i \cos 3a_k},$$

where i, k are all duads from $1, 2 \ldots n$.

5. Returning to cubic determinants, we can represent the cubic determinant by either of the products

$$\Pi\,(b_{i1} e_1 + b_{i2} e_2 + \ldots + b_{in} e_n),$$

$$\Pi\,(c_{i1} \epsilon_1 + c_{i2} \epsilon_2 + \ldots + c_{in} \epsilon_n),$$

where

$$b_{ik} = a_{i1k} \epsilon_1 + a_{i2k} \epsilon_2 + \ldots + a_{ink} \epsilon_n,$$

$$c_{ik} = a_{ik1} e_1 + a_{ik2} e_2 + \ldots + a_{ikn} e_n.$$

From the first form we see that, by interchanging two strata, the determinant undergoes a change of sign, as belonging to the system e; from the second, that it undergoes a change of sign, as belonging to the system ϵ. Thus interchanging two strata leaves the determinant unaltered.

Interchanging a pair of planes of either of the other systems is equivalent to interchanging two units of the system e or ϵ, and hence changes the sign of the determinant.

For a determinant of class p, we should see that interchanging two suffixes of a system which is permutated, changes the sign of the determinant; but the interchange of two of the fixed suffixes introduces

$p-1$ changes of sign, and hence leaves the determinant unaltered if p is odd.

If all the strata of a cubic determinant become identical, we may suppress the first suffix, and regard the determinant as formed by repeating the array

$$\begin{matrix} a_{11} & \dots & a_{1n}, \\ \dots & & \dots \\ a_{n1} & \dots & a_{nn}, \end{matrix}$$

n times. The determinant is then equal to

$$(e_1 A_1 + e_2 A_2 + \dots + e_n A_n)^n,$$

where $A_i = a_{i1}\epsilon_1 + a_{i2}\epsilon_2 + \dots + a_{in}.$

And this $= n!\, e_1 e_2 \dots e_n \,.\, A_1 A_2 \dots A_n$

$$= n! \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

Similarly, if all systems of elements of a determinant of order n, which have the same first suffix, become identical, the determinant reduces to

$n! \times$ determinant of next lower class,

if the original determinant was of odd class, but vanishes if its class was even.

6. A cubic determinant of order n is the sum of $n!$ ordinary determinants of order n. For, taking the second representation of Art. 5,

$$\begin{aligned} A &= | a_{ijk} | \\ &= \Pi \left(c_{i1}\epsilon_1 + c_{i2}\epsilon_2 + \dots + c_{in}\epsilon_n \right). \end{aligned}$$

And this product gives the ordinary determinant

$$| c_{ik} |,$$

whose elements are alternate numbers of the n^{th} order. To split this determinant up into others with simple elements, we must take a partial column from each column of the determinant; but if we take a partial column in the p^{th} place from one column, we must not take a partial column in the p^{th} place from any other column, for then e_p would occur twice, and the corresponding determinant would vanish. Hence each selection of partial columns must have for the suffixes of its units a permutation of 1, 2 ... n. There are $n!$ such, and as many determinants. Hence

$$A = \Sigma \,| a_{ij(k)} |,$$

where, in the determinant on the right, k is put in brackets to remind us that, though it varies from one column to another, in the same determinant it remains fixed, while i, j take all values from 1, 2 ... n.

7. The product of two ordinary determinants of order n can be represented as a cubic determinant of order n. For, let

$$A = | a_{ik} | = A_1 A_2 \dots A_n,$$
$$B = | b_{ik} | = B_1 B_2 \dots B_n$$

be the determinants, where

$$A_i = a_{i1}\epsilon_1 + a_{i2}\epsilon_2 + \dots + a_{in}\epsilon_n,$$
$$B_i = b_{i1}e_1 + b_{i2}e_2 + \dots + b_{in}e_n.$$

Then $$AB = \Pi A_i B_i,$$

the general term in the product $A_i B_i$ is

$$a_{ij} b_{ik} \epsilon_j e_k;$$

or, if $$c_{ijk} = a_{ij} b_{ik},$$

we see that $$AB = | c_{ijk} | \quad (i, j, k = 1, 2 \dots n).$$

By multiplying A_i and B_i together, we avoid any change of sign arising from inversion of the units e or ϵ among themselves. If we allow for the change of sign, we get numerous other representations of the product AB in the form of a cubic determinant.

8. The product of a cubic determinant A whose elements are a_{ijk}, and of an ordinary determinant B whose elements are b_{ik}, can be represented either as a determinant of the fourth class whose elements are given by

$$c_{ijkl} = a_{ijk} b_{il},$$

or as a cubic determinant whose elements are given by

$$c_{ijk} = b_{j1} a_{ik1} + b_{j2} a_{ik2} + \dots + b_{jn} a_{ikn}.$$

To prove the first part of this theorem, we have only to express the determinants in the form of products, viz.,

$$A = \Pi A_i, \quad B = \Pi B_i;$$

therefore $$AB = \Pi A_i B_i,$$

and the general term in the product on the right is

$$a_{ijk} b_{il} e_j \epsilon_k \eta_l.$$

Thus $$AB = | c_{ijkl} | \quad (i, j, k, l = 1, 2 \dots n).$$

For the second part of the theorem, we have

$$\begin{aligned} C = \Pi\,(&c_{i11}\epsilon_1 e_1 + c_{i12}\epsilon_1 e_2 + \ldots + c_{i1n}\epsilon_1 e_n \\ &+ c_{i21}\epsilon_2 e_1 + c_{i22}\epsilon_2 e_2 + \ldots + c_{i2n}\epsilon_2 e_n \\ &+ \ldots \\ &+ c_{in1}\epsilon_n e_1 + c_{in2}\epsilon_n e_2 + \ldots + c_{inn}\epsilon_n e_n) \\ = \Pi\,(&a_{i11}B_1 e_1 + a_{i12}B_1 e_2 + \ldots + a_{i1n}B_1 e_n \\ &+ a_{i21}B_2 e_1 + a_{i22}B_2 e_2 + \ldots + a_{i2n}B_2 e_n \\ &+ \ldots \\ &+ a_{in1}B_n e_1 + a_{in2}B_n e_2 + \ldots + a_{inn}B_n e_n), \end{aligned}$$

where $$B_i = b_{i1}\epsilon_1 + b_{i2}\epsilon_2 + \ldots + b_{in}\epsilon_n.$$

Since the alternate numbers B_i follow the same laws as the units, the last product is the representation of the determinant A by means of the numbers $e_1 \ldots e_n$, $B_1 \ldots B_n$; and hence

$$\begin{aligned} C &= Ae_1 \ldots e_n B_1 \ldots B_n \\ &= AB. \end{aligned}$$

9. Exactly similar reasoning shows that the product of two cubic determinants A and B, whose elements are a_{ijk} and b_{ijk}, both of order n, can be represented as a determinant of class five and order n, whose elements are given by $$c_{ipqrs} = a_{ipq}b_{irs};$$

or as a determinant of class four and order n, whose elements are given by $$c_{pqrs} = \Sigma a_{ipq}b_{irs} \quad (i = 1, 2 \ldots n).$$

And, in general, for two determinants of the same order. The product of two determinants of classes p and q, whose elements are

$$a_{ij\ldots k},\ b_{ij\ldots l}$$

respectively, can be represented either as a determinant of class $p+q-1$, whose elements are given by

$$c_{ij\ldots kuv\ldots s} = a_{ij\ldots k}b_{iuv\ldots s},$$

or as a determinant of class $p+q-2$, whose elements are given by

$$c_{j\ldots ku\ldots s} = \Sigma a_{ij\ldots k}b_{iu\ldots s} \quad (i = 1, 2 \ldots n).$$

10. The product of any number of determinants can also be represented as a determinant. For example, the product of the three determinants $A = |a_{ik}|$, $B = |b_{ik}|$, $C = |c_{ik}|$ $(i, k = 1, 2 \ldots n)$ can be represented by the determinant of the fourth class

$$|(ipqr)| \quad (i, p, q, r = 1, 2 \ldots n),$$

where $$(ipqr) = a_{ip}b_{iq}c_{ir}.$$

And, in general, the product of x determinants of order n and classes $p, q, r \ldots$, can be represented as a determinant of order n and class $p+q+r+\ldots-x+1$.

11. For two pairs of points in a straight line, we have the well-known relation

$$AB\,.\,CD+AC\,.\,DB+AD\,.\,BC=0.$$

To which we can give the form

$$\begin{vmatrix} AC, & AD \\ BC, & BD \end{vmatrix} = AB\,.\,CD;$$

or, for two sets of three points, A, B, C; D, E, F, we have

$$\begin{vmatrix} AD, & AE, & AF \\ BD, & BE, & BF \\ CD, & CE, & CF \end{vmatrix} = 0.$$

This is really identical with the former equation, as we see by expanding with regard to the elements of the first row. Also, if

$$\begin{array}{l} u_1,\ u_2\ \ldots, \\ v_1,\ v_2\ \ldots \end{array}$$

be two sets of binary quantics, and if J_{ik} is the Jacobian of u_i and v_k, we see, by compounding the arrays

$$\begin{array}{cc} \dfrac{du_1}{dx}, & \dfrac{du_1}{dy} \\ \dfrac{du_2}{dx}, & \dfrac{du_2}{dy} \\ \ldots & \ldots \end{array} \quad\text{and}\quad \begin{array}{cc} \dfrac{dv_1}{dy}, & -\dfrac{dv_1}{dx}, \\ \dfrac{dv_2}{dy}, & -\dfrac{dv_2}{dx}, \\ \ldots & \ldots \end{array}$$

that

$$\begin{vmatrix} J_{11}, & J_{12}, & J_{13} \\ J_{21}, & J_{22}, & J_{23} \\ J_{31}, & J_{32}, & J_{33} \end{vmatrix} = 0,$$

$$\begin{vmatrix} J_{11}, & J_{12} \\ J_{21}, & J_{22} \end{vmatrix} = \frac{d\,(u_1,\,u_2)}{d\,(x,\,y)} \cdot \frac{d\,(v_1,\,v_2)}{d\,(x,\,y)}.$$

I propose to establish some general theorems, of which these are particular cases.

12. If three sets, each of three points, are given in a plane, and each set of points numbered 1, 2, 3; then

$$\Delta_{ijk}$$

is taken to mean the area of the triangle formed by the three points, i of the first, j of the second, and k of the third system. This being so, the cubic determinant

$$|\,\Delta_{ijk}\,|$$

vanishes. For we know that it changes sign if we interchange two second or two third suffixes, and hence vanishes identically if these coincide, but does not change sign if we interchange two first suffixes. It follows that the determinant contains as factors Δ_2 and Δ_3, the areas of the triangles formed by the second and third systems of points; and the consideration of dimension shows that the remaining factor is independent of these two systems.

Now, if we interchange the second and third systems of points, we only change the sign of each element of the determinant without altering its value (the triangles ABC and ACB being supposed of equal magnitude but opposite signs); this alteration changes the sign of the determinant, but only alters the order of the factors Δ_2 and Δ_3. Hence the determinant must vanish. Hence, for three systems of points in a plane,

$$|\Delta_{ik}| = 0 \quad (i, j, k = 1, 2, 3).$$

13. Four sets of points in space being numbered in like manner, we can form a determinant of the fourth class with the volumes of the tetrahedra obtained by taking a point from each system. The volume of the tetrahedron formed by the points i, j, k, l of the four systems being denoted by

$$V_{ijkl}.$$

I shall show that

$$|V_{ijkl}| = -4!\,V_1V_2V_3V_4 \quad (i, j, k, l = 1, 2, 3, 4),$$

where V_1, V_2, V_3, V_4 are the volumes of the tetrahedron formed by the points of each system separately.

For our determinant, being of even class, changes sign if we interchange any two points in one of the systems, and hence vanishes if two points of the same system coincide. Hence it must be divisible by the product

$$V_1V_2V_3V_4,$$

and is equal to this multiplied by a numerical factor.

To determine the factor, let all the systems coincide with the first. Then, if we consider the expression for the determinant as a product of alternate numbers, the first factor will be

$$V_{1ijk}e_i\epsilon_j\eta_k,$$

where i, j, k is a permutation of 2, 3, 4.

If, then, the volume of the tetrahedron formed by the four points be denoted by V, this first factor is

$$V\begin{vmatrix} e_2, & e_3, & e_4 \\ \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \eta_2, & \eta_3, & \eta_4 \end{vmatrix} = V(234).$$

And the value of the determinant is given by

$$\begin{aligned} D &= V^4 (234)(341)(412)(123) \\ &= -4!\, V^4, \end{aligned}$$

by a theorem of Clifford's, given by Mr. Spottiswoode, "Proceedings of London Mathematical Society," Vol. vii., p. 112.

14. There is no difficulty in extending the theorems of Arts. 12 and 13 to any number of dimensions, regarded either as algebraic theorems or as geometric relations. We can, in fact, enuntiate the general theorem. If

$$D_1, D_2 \ldots D_p$$

are p determinants of class 2 (*i.e.*, ordinary determinants), and if

$$(ij \ldots k)$$

denotes the ordinary determinant obtained by taking for its first row the i^{th} row of D_1, for its second the j^{th} row of D_2, &c., then the determinant of class p and order p,

$$|\,(ij \ldots k)\,| \quad (i, j \ldots k = 1, 2 \ldots p),$$

vanishes if p is odd, but if p is even and greater than 2, its value is

$$-p!\, D_1 D_2 \ldots D_p.$$

By allowing the identity for $2p+1$ points to degrade, by supposing all the points of the first system to coincide, while all the points of the other systems lie in the same space of $2p-1$ dimensions, we get another relation for magnitudes of odd dimensions.

For example, if, in the relation of Art. 12, all the points of the first system coincide, the determinant by Art. 5 reduces to six times the ordinary determinant

$$\begin{vmatrix} (111), & (112), & (113) \\ (121), & (122), & (123) \\ (131), & (132), & (133) \end{vmatrix} = 0.$$

If we allow the points of the first system to coincide with one of the vertices of the fundamental triangle, while the points of the other systems lie on the opposite side, we regain the relation of Art. 11.

Similarly, for five sets of points in four-dimensional space, we have

$$|\,(p, q, r, u, v)\,| = 0.$$

Now, let all the points of the first system coincide, while the other four sets of five points are taken in the same space of three dimensions, and we get the new relation

$$|\,V_{qruv}\,| = 0 \quad (q, r, u, v = 1, 2 \ldots 5)$$

between the volumes of the tetrahedra formed by four systems, each of five points.

If our determinants, instead of representing the volumes of geometric figures, are taken to represent other magnitudes, we get numerous other theorems. For example, for three systems of straight lines passing through the same point, if

$$\sin(ijk)$$

is the sine of the solid angle formed by the lines i, j, k of the three systems,

$$|\sin(ijk)| = 0 \quad (i, j, k = 1, 2, 3).$$

For three sets of ternary quantics

$$\begin{matrix} u_1, & u_2 \; \ldots, \\ v_1, & v_2 \; \ldots, \\ w_1, & w_2 \; \ldots, \end{matrix}$$

if

$$J(i, j, k) = \frac{d(u_i, v_j, w_k)}{d(x, y, z)},$$

then

$$|J(i, j, k)| = 0 \quad (i, j, k = 1, 2, 3),$$

and so on.

15. Mr. Spottiswoode (*l. c.*) has given some theorems concerning determinants of alternate numbers, incorporating with his own researches some of the late Professor Clifford. In the determinants there considered, the elements are simple alternate numbers; the general theorem of the preceding article can be regarded as a theorem concerning determinants whose elements are products of alternate numbers. Then, in fact,

$$D_1, D_2 \ldots D_p^n$$

stand for p sets, each of p alternate numbers, belonging to the same system of units of the p^{th} order, and the elements $(i, j \ldots k)$ of our determinant are products formed by selecting factors from each of these p sets. This leads us to consider determinants whose elements are, binary, ternary, &c., products of alternate numbers, belonging to independent systems.

Let

$$\begin{matrix} \alpha_1, & \alpha_2, & \alpha_3 \; \ldots, \\ \beta_1, & \beta_2, & \beta_3 \; \ldots \end{matrix}$$

be two independent systems of alternate numbers of the n^{th} order. Then the determinant

$$\begin{vmatrix} (\alpha_1\beta_1), & (\alpha_1\beta_2), & (\alpha_1\beta_3) & \ldots \\ (\alpha_2\beta_1), & (\alpha_2\beta_2), & (\alpha_2\beta_3) & \ldots \\ (\alpha_3\beta_1), & (\alpha_3\beta_2), & (\alpha_3\beta_3) & \ldots \\ \ldots & \ldots & \ldots & \ldots \; \ldots \end{vmatrix}$$

vanishes if its order is greater than n; but if its order is equal to n, its value is

$$n!\, \alpha_1 \alpha_2 \dots \alpha_n \,.\, \beta_1 \beta_2 \dots \beta_n.$$

The first part is evident, for each term of the determinant will contain a product of more than n numbers of the same system, and hence vanishes. For the second part, consider the term

$$(\alpha_1 \beta_p)(\alpha_2 \beta_q)(\alpha_3 \beta_r)\dots,$$

or

$$\alpha_1 \alpha_2 \alpha_3 \dots \beta_p \beta_q \beta_r \dots,$$

the sign of this term is $(-1)^k$, where k is the number of inversions in the permutation

$$p, q, r \dots;$$

but if we alter the order of the factors so as to bring this to the order 1, 2, 3 ..., we introduce k changes of sign; thus all the terms are positive, and the theorem follows.

For the three independent systems of the n^{th} order

$$\alpha_1,\ \alpha_2,\ \alpha_3 \dots,$$
$$\beta_1,\ \beta_2,\ \beta_3 \dots,$$
$$\gamma_1,\ \gamma_2,\ \gamma_3 \dots,$$

the cubic determinant whose elements are

$$(\alpha_i \beta_j \gamma_k)$$

vanishes, if its order is greater than n; but if it is of the n^{th} order, its value is

$$\{n!\}^2 \alpha_1 \alpha_2 \dots \alpha_n \,.\, \beta_1 \beta_2 \dots \beta_n \,.\, \gamma_1 \gamma_2 \dots \gamma_n,$$

and so on for determinants of higher class.

If the alternate numbers are the factors of the ordinary determinants

$$A = |a_{ik}|,\quad B = |b_{ik}|,\quad C = |c_{ik}|,$$

we obtain new representations for the products AB, ABC, &c.

Notes on a Class of Definite Integrals. *By the* Rev. T. R. TERRY, M.A., Fellow of Magdalen College, Oxford.

[*Read Nov.* 13*th*, 1879.]

1. The integrals considered in the following paper are of the forms

$$\int_0^\pi \frac{\cos px\, dx}{(1-2a\cos x+a^2)^n} \text{ and } \int_0^\pi \frac{\sin^{2p} x\, dx}{(1-2a\cos x+a^2)^{n+p}},$$

where p is a positive integer, and n any real quantity, positive or negative, integral or fractional.

The values of these integrals, when $a>1$, may be readily deduced from the values when $a<1$. It is sufficient, therefore, to consider the case when $a<1$. For the sake of brevity, write

$$\Delta = 1-2a\cos x+a^2.$$

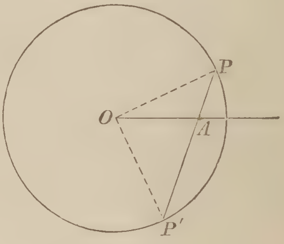
2. In a paper in the "Journal de l'École Polytechnique," tom. x., Poisson has considered the integral $\int_0^\pi \frac{\sin^m x\,dx}{\Delta^n}$, and, by means of the differential equation which it satisfies, has shown that

$$\int_0^\pi \frac{\sin^m x\,dx}{\Delta^n} = (1-a^2)^{1+m-2n}\int_0^\pi \frac{\sin^m x\,dx}{\Delta^{1+m-n}},$$

where the only restriction on m and n is that $m+1$ must be positive.

The following proof of this theorem is a simple extension of the method used in the solution of one of Mr. Ferrers' questions in the "Solutions of Cambridge Problems and Riders for 1878" (pp. 114, 115).

Let O be the centre of a circle whose radius is unity, and A a fixed point within it, PAP' any chord through A.



Let $OA = a$, $POA = \theta$, $P'OA = \phi$, and

$$\Delta = PA^2 = 1-2a\cos\theta+a^2,$$

$$\Delta' = P'A^2 = 1-2a\cos\phi+a^2,$$

then $$\Delta\Delta' = (1-a^2)^2.$$

If the chord PAP' turns through a small angle, so as to come into the position pAp', we have

$$d\theta = Pp,\ d\phi = -P'p',\ \text{also}\ \frac{Pp}{P'p'} = \frac{PA}{P'A} = \frac{1-a^2}{\Delta'};$$

therefore $$d\theta = -\frac{1-a^2}{\Delta'}\,d\phi.$$

Again, if perpendiculars be drawn from P and P' on OA,

$$\frac{\sin\theta}{\sin\phi} = \frac{\text{perp. from } P}{\text{perp. from } P'} = \frac{PA}{P'A} = \frac{1-a^2}{\Delta'}.$$

Hence, by substituting, and integrating between limits 0 to π for θ, which correspond respectively to limits π to 0 for ϕ, we have

$$\int_0^\pi \frac{\sin^m\theta\,d\theta}{(1-2a\cos\theta+a^2)^n} = (1-a^2)^{1+m-2n}\int_0^\pi \frac{\sin^m\phi\,d\phi}{(1-2a\cos\phi+a^2)^{1+m-n}},$$

which is the required result. It is clear that $m+1$ must be positive, otherwise each integral would become infinite. The theorem is true if $m=0$, and it is this particular case which is proved in the "Solutions" above referred to.

3. In Bertrand's "Calcul Intégral," pp. 173, 174, it is proved that

$$\int_0^\pi \Phi^p(\cos x)\sin^{2p}x\,dx = \frac{(2p)!}{2^p\,.\,p!}\int_0^\pi \Phi(\cos x)\cos px\,dx,$$

where p is any positive integer, $\Phi(\cos x)$ any function of $\cos x$, and $\Phi^p(\cos x)$ its p^{th} differential with respect to $\cos x$. This theorem, as is well-known, is due to Jacobi ("Crelle," Bd. xv., p. 3).

4. Let us now consider the expansion

$$\Delta^{-n} = A_0 + 2A_1\cos x + \ldots + 2A_p\cos px + \ldots,$$

where n is any real quantity, and p any positive integer.

Then ("Gauss, Werke," Bd. iii., pp. 128, 129) A_p may be expressed in four ways by means of the hypergeometric series.

First, if $r = \cos x + i\sin x$, we have

$$\Delta^{-n} = (1-ar)^{-n}(1-ar^{-1})^{-n};$$

hence, expanding and equating coefficients of $\cos px$, we get

$$A_p = \frac{n(n+1)\ldots(n+p-1)}{p!}a^p F\{n+p,\ n,\ p+1,\ a^2\}.$$

Secondly, by putting

$$\Delta^{-n} = \frac{1}{(1+a^2)^n}\left\{1-\frac{a}{1+a^2}(r+r^{-1})\right\}^{-n},$$

and proceeding as before, we have

$$A_p = \frac{n(n+1)\ldots(n+p-1)}{p!}\ \frac{a^p}{(1+a^2)^{n+p}}$$
$$\times F\left\{\frac{n+p}{2},\ \frac{n+p+1}{2},\ p+1,\ \frac{4a^2}{(1+a^2)^2}\right\}.$$

Thirdly, by putting

$$\Delta^{-n} = \frac{1}{(1+a)^{2n}}\left\{1-\frac{a}{(1+a)^2}(r^{\frac{1}{2}}+r^{-\frac{1}{2}})^2\right\}^{-n},$$

we have $A_p = \dfrac{n(n+1)\ldots(n+p-1)}{p!}\ \dfrac{a^p}{(1+a)^{2n+2p}}$

$$\times F\left\{n+p,\ \frac{2p+1}{2},\ 2p+1,\ \frac{4a}{(1+a)^2}\right\}.$$

Fourthly, by putting

$$\Delta^{-n} = \frac{1}{(1-a)^{2n}} \left\{ 1 + \frac{a}{(1-a)^2} \left(r^{\frac{1}{2}} - r^{-\frac{1}{2}}\right)^2 \right\}^{-n},$$

we have $A_p = \dfrac{n(n+1)\ldots(n+p-1)}{p\,!} \, \dfrac{a^p}{(1-a)^{2n+2p}}$

$$\times F\left\{ n+p,\ \frac{2p+1}{2},\ 2p+1,\ \frac{-4a}{(1-a)^2} \right\}.$$

If $-n$ is a positive integer less than p, then obviously $A_p = 0$. Now, multiplying both sides of the expression by $\cos px$, and integrating with respect to x from 0 to π, we have

$$\int_0^\pi \frac{\cos px\, dx}{\Delta^n} = \pi A_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

5. In Jacobi's theorem, put $\Phi(\cos x) = \Delta^{-n}$, then

$$\Phi^p(\cos x) = \frac{n(n+1)\ldots(n+p-1)}{\Delta^{n+p}} (2a)^p.$$

[We may notice in passing that, if $-n$ is a positive integer less than p, the right-hand side of this equation vanishes.]

Hence, by means of (1), we have

$$\int_0^\pi \frac{\sin^{2p} x\, dx}{\Delta^{n+p}} = \frac{1}{(2a)^p} \frac{(2p)\,!}{2^p \,.\, p\,!} \frac{\pi A_p}{n(n+1)\ldots(n+p-1)} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

$$= \pi \frac{(2p)\,!}{2^{2p} \,.\, p\,!\, p\,!} F\{n+p,\ n,\ p+1,\ a^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

$$= \pi \frac{(2p)\,!}{2^{2p} \,.\, p\,!\, p\,!} \frac{1}{(1+a^2)^{n+p}} F\left\{ \frac{n+p}{2},\ \frac{n+p+1}{2},\ p+1,\ \frac{4a^2}{(1+a^2)^2} \right\} \ldots(\beta),$$

$$= \pi \frac{(2p)\,!}{2^{2p} \,.\, p\,!\, p\,!} \frac{1}{(1+a)^{2n+2p}} F\left\{ n+p,\ \frac{2p+1}{2},\ 2p+1,\ \frac{4a}{(1+a)^2} \right\} \ldots(\gamma),$$

$$= \pi \frac{(2p)\,!}{2^{2p} \,.\, p\,!\, p\,!} \frac{1}{(1-a)^{2n+2p}} F\left\{ n+p,\ \frac{2p+1}{2},\ 2p+1,\ \frac{-4a}{(1-a)^2} \right\} \ldots(\delta).$$

6. We must first consider the case in which the proof of the formulæ (2) fails, *i.e.*, when $-n$ is a positive integer $< p$. As the resulting formulæ have no critical values in this case, and as, by giving suitable values to n, we may make $-n$ approach as near as we please on either side to any positive integer $< p$, we see, by the principle of continuity, that the formulæ must be true also in this case.

A rigorous proof may be given as follows :—

By Poisson's theorem, we have

$$\int_0^\pi \frac{\sin^{2p} x\,dx}{\Delta^{n+p}} = (1-a^2)^{1-2n} \int_0^\pi \frac{\sin^{2p} x\,dx}{\Delta^{p+1-n}}.$$

Since to the second integral the method certainly applies when $-n$ is a positive integer $< p$, we have, by means of (2),

$$\int_0^\pi \frac{\sin^{2p} x\,dx}{\Delta^{n+p}} = (1-a^2)^{1-2n} \cdot \pi \frac{(2p)\,!}{2^{2p} \cdot p\,!\,p\,!}\, F\{p+1-n,\ 1-n,\ p+1,\ a^2\}.$$

But Gauss has proved (Werke, Bd. iii., p. 209) that

$$F\{\alpha, \beta, \gamma, x\} = (1-x)^{\gamma-\alpha-\beta} F\{\gamma-\beta, \gamma-\alpha, \gamma, x\}.$$

Hence $$\int_0^\pi \frac{\sin^{2p} x\,dx}{\Delta^{n+p}} = \pi \frac{(2p)\,!}{2^{2p} \cdot p\,!\,p\,!}\, F\{n+p, n, p+1, a^2\},$$

which is the required result.

7. The four forms of the result (2) were obtained by using different expansions for the same expression. Their equivalence may be proved directly. Kummer ("Crelle," Bd. xv., p. 78) has proved that

$$F\{\alpha, \beta, \alpha-\beta+1, x\} = (1+x)^{-\alpha} F\left\{\frac{\alpha}{2}, \frac{\alpha+1}{2}, \alpha-\beta+1, \frac{4x}{(1+x)^2}\right\},$$

and also $$= (1 \pm \sqrt{x})^{-2\alpha} F\left\{\alpha, \alpha-\beta+\tfrac{1}{2}, 2\alpha-2\beta+1, \frac{\pm 4\sqrt{x}}{(1 \pm \sqrt{x})^2}\right\};$$

and, by putting $\alpha = n+p$, $\beta = n$, $\gamma = p+1$, $x = a^2$, we see that the first of the forms in (2) is equivalent to any of the others.

8. Legendre (Exerc., tom. 1, pp. 372–6) has considered the integrals

$$P_n = \int_0^\pi \frac{\cos px}{\Delta^n}\,dx, \text{ and } Q_n = \int_0^\pi \Delta^n \cos px\,dx,$$

where p and n are positive integers. The first of these he evaluated by the method of induction, the second by expanding Δ^n directly. The method used in the second case would have applied equally well to the first case.

The values which he obtained are

$$P_n = \frac{\pi a^p}{(1-a^2)^{2n-1}} \cdot \frac{(p+1)(p+2)\ldots(p+n-1)}{1\,.\,2\ldots(n-1)}$$

$$\times \left\{1+\frac{n-1}{1} a^2 \frac{n-p-1}{p+1} + \frac{(n-1)(n-2)}{1\,.\,2} a^4 \frac{(n-p-1)(n-p-2)}{(p+1)(p+2)} + \ldots\right\},$$

$$Q_n = \pi\,(-a)^p\,.\,\frac{n\,(n-1)\,\dots\,(n+1-p)}{1\,.\,2\dots p}$$

$$\times\left\{1+\frac{n}{1}\,a^2\,\frac{n-p}{p+1}+\frac{n\,(n-1)}{1\,.\,2}\,a^4\,\frac{(n-p)\,(n-p-1)}{(p+1)\,(p+2)}+\dots\right\}.$$

Thus, we see at once that

$$Q_n = \pi\,\frac{(-n)\,(-n+1)\,\dots\,(-n+p-1)}{p\,!}\,a^p F\{-n+p,\ -n,\ p+1,\ a^2\},$$

which agrees with (1) when we write $-n$ for n in that formula.

Again,

$$P_n = \pi\,\frac{n\,(n+1)\,\dots\,(n+p-1)}{p\,!}\,\frac{a^p}{(1-a^2)^{2n-1}}\,F\{1+p-n,\ 1-n,\ p+1,\ a^2\}.$$

But, since ("Gauss, Werke," Bd. iii., p. 209)

$$F\{\alpha,\ \beta,\ \gamma,\ x\} = (1-x)^{\gamma-\alpha-\beta}\,F\{\gamma-\beta,\ \gamma-\alpha,\ \gamma,\ x\},$$

we have $P_n = \pi\,\dfrac{n(n+1)\dots(n+p-1)}{p\,!}\,a^p\,F\{n+p,\ n,\ p+1,\ a^2\}$,

which also agrees with (1).

9. We must next consider the formula given by Jacobi. In "Crelle," Bd. xv., p. 11, Jacobi proved, by a rather long process, that

$$\int_0^{\pi}\frac{\sin^{2p}x\,dx}{\Delta^{n+p}} = \frac{1\,.\,3\,\dots\,(2p-1)}{2\,.\,4\,\dots\,2p}\,.\,\frac{\pi}{(1-a^2)^n}$$

$$\times\left\{1+\frac{(n-1)\,n}{1\,.\,(p+1)}\,\frac{a^2}{1-a^2}+\frac{(n-2)\,(n-1)\,n\,(n+1)}{1\,.\,2\,(p+1)\,(p+2)}\,\frac{a^4}{(1-a^2)^2}+\dots\right\}.$$

The proof applies for all values of n, except $n=\frac{1}{2}$, which case he had discussed in the earlier part of the same paper. Using the hypergeometric series, this result may be expressed in the form

$$\frac{(2p)\,!}{2^{2p}\,.\,p\,!\,p\,!}\,\frac{\pi}{(1-a^2)^n}\,F\left\{n,\ 1-n,\ p+1,\ \frac{a^2}{a^2-1}\right\}.$$

But Kummer has proved ("Crelle," Bd. xv., p. 52) that

$$F\{\alpha,\ \beta,\ \gamma,\ x\} = (1-x)^{-\beta}\,F\left\{\beta,\ \gamma-\alpha,\ \gamma,\ \frac{x}{x-1}\right\}.$$

Thus, Jacobi's result becomes

$$\pi\,\frac{(2p)\,!}{2^{2p}\,.\,p\,!\,p\,!}\,F\{n+p,\ n,\ p+1,\ a^2\},$$

which agrees with (2).

10. By integration by parts, we have

$$\int_0^\pi \frac{\sin^{2p} x \cos x \, dx}{\Delta^{n+p}} = \frac{2a\,(n+p)}{2p+1} \int_0^\pi \frac{\sin^{2p+2} x \, dx}{\Delta^{n+p+1}};$$

which, by (2),

$$= \frac{2a\,(n+p)}{2p+1} \cdot \pi \frac{(2p+2)\,!}{2^{2p+2}\,.\,(p+1)\,!\;(p+1)\,!} F\{n+p+1,\, n,\, p+2,\, a^2\}$$

$$= \pi a \frac{(n+p)}{p+1} \cdot \frac{(2p)\,!}{2^{2p}\,.\,p\,!\;p\,!} F\{n+p+1,\, n,\, p+2,\, a^2\}.$$

Note on a Problem of Fibonacci's. *By* SAMUEL ROBERTS, F.R.S.

[*Read Nov.* 13*th*, 1879.]

1. If the equations

$$x^2 + Py^2 = u^2, \quad x^2 - Py^2 = v^2$$

coexist in integers, $\pm Py^2$ must be of the form $4ab\,(a^2-b^2)$, a and b being of different species, or of the form $ab\,(a^2-b^2)$, a and b being of the same species. In fact, u and v are both even or both odd; and, if we write $\alpha+\beta$ for u, $\alpha-\beta$ for v, then

$$x^2 = \alpha^2+\beta^2, \quad Py^2 = 2\alpha\beta,$$

and, putting a^2-b^2 for α, $2ab$ for β, we get

$$x = a^2+b^2, \quad Py^2 = 4ab\,(a^2-b^2),$$

where a, b may be taken as of different species. On the other hand, if a, b are of the same species, x^2, Py^2, u^2, v^2 are all divisible by 4, so that $Py'^2 = ab\,(a^2-b^2)$. Otherwise; supposing a, b to be of different species, we may write a' for $a+b$, b' for $a-b$, so that

$$Py^2 = a'b'\,(a'^2-b'^2), \quad x = \left(\frac{a'+b'}{2}\right)^2 + \left(\frac{a'-b'}{2}\right)^2,$$

$$u = a'b' + \frac{a'^2-b'^2}{2}, \quad v = a'b' - \frac{a'^2-b'^2}{2}.$$

Another form is given by

$$P\,.\,4x^2y^2 = u^4 - v^4;$$

and, generally, taking $\alpha^2\beta^2\,(\alpha^4-\beta^4)$ for $ab\,(a^2-b^2)$, it appears that

$$\pm P = \frac{\alpha^4-\beta^4}{q^2};$$

where q^2 may or may not be the greatest square factor in $\alpha^4-\beta^4$, and α, β are, as we please, of the same or different species. In this case, we have the identity

$$(a^4+\beta^4)^2 \pm \left(\frac{a^4-\beta^4}{q^2}\right).4a^2\beta^2.q^2 = (a^4-\beta^4 \pm 2a^2\beta^2)^2;$$

the ambiguous signs being taken both plus or both minus. We do not, however, really limit the problem if we consider P as free from square factors, and a, b as prime to one another, and of different species. The form of P is then given by

$$ab(a^2-b^2) = \pm P.V^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

where V^2 is the greatest square which is a factor of the left-hand expression.

It is easy, therefore, to obtain an indefinite number of admissible values of P; but if we aim at obtaining primes, or the double of primes, the direct method becomes impracticable.*

A more manageable rule, employed by Leonardo Fibonacci, consisted in making three of the numbers a, b, $a-b$, $a+b$, squares. This rule includes all admissible primes. I have thought it worth while to develope the analysis a little further than I find it carried in the work of Prof. Genocchi, referred to below, in order to narrow the field and obtain more frequent prime results.

2. For prime values of P, the cases are as follows:—

(I.) $a = k^2,\ b = l^2,\ k^2+l^2 = m^2,\ k^2-l^2 = \pm Pn^2;$

whence we have $k = p^2-q^2,\ l = 2pq;$

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* Professor Genocchi gives a full account, with further developments of Fibonacci's investigations, in a memoir entitled, "Sopra Tre Scritti inediti di Leonardo Pisano pubblicati da Baldasarre Boncompagni Note Analitiche" (Tortolini, Ann., Tom. vi., 1855). It appears that the "Liber Quadratorum" of the Pisan Mathematician originated in his study of the system $x^2+5 = y^2$, $x^2-5 = z^2$.

Numbers of the form $4ab(a^2-b^2)$, when a and b are of different species, and $ab(a^2-b^2)$, when a and b are of the same species, are termed by him congruous numbers; and the connected rational squares are said to be congruent to them. Their theory is presented in an ingenious form, dependent on the theorem that the sum of consecutive odd numbers commencing with unity is a square.

The general system of equations has been since treated of by Pacioli (the author of the first printed Algebra), Cossali, and others. Euler notices it briefly in his "Algebra."

The evaluation of x, y, u, v is easy, when P is known in the form of $\frac{ab(a^2-b^2)}{q^2}$ or $\frac{\alpha^4-\beta^4}{q^2}$. The difficulty of the problem consists in determining whether a given number P is admissible; or, as it may be stated, whether three rational squares in arithmetical progression can have P as their common difference. When one solution x, y, u, v is known, an infinite number of other solutions can be found. Fibonacci, who wrote in the first half of the thirteenth century, gives the theorem that P cannot be a square; in other words, that the difference of two different fourth powers cannot be a square.

I ought also to refer to a paper by M. Ed. Lucas, "Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'Arithmétique Supérieure" (Boncompagni, Bull. x., pp. 129—193, 239—293, 1877). The part of the memoir bearing on the present subject is, however, brief. M. Lucas shows how to obtain the systems of solutions (x, y, u, v), exemplifying his method by the cases of $P = 5$, $P = 6$.

and
$$p^2-q^2+2pq = \pm Pr^2,$$
$$p^2-q^2-2pq = \pm s^2.$$

If p and q be taken negatively as well as positively, these equations comprise all the necessary forms. Moreover, we need only take s^2 with the positive sign, because

$$\left.\begin{array}{l}(p+q)^2-2q^2 = \pm Pr^2\\(p-q)^2-2q^2 = s^2\end{array}\right\} \text{ and } \left\{\begin{array}{l}(p-q)^2-2p^2 = \pm Pr^2\\(p+q)^2-2p^2 = s^2\end{array}\right.$$

are equivalent systems.

Putting, then, $s = 2\lambda^2-\mu^2$, which is its necessary form, we have

$$s^2 = 2\,\{2\lambda^2+\mu^2\pm 2\lambda\mu\}^2-(2\lambda^2+\mu^2\pm 4\lambda\mu)^2,$$

and
$$p = 2\lambda^2+\mu^2\pm 2\lambda\mu,$$
$$p+q = 2\lambda^2+\mu^2\pm 4\lambda\mu\,;$$

the ambiguous signs being taken both positive or both negative. Consequently
$$\{2(\lambda+\mu)^2-\mu^2\}^2-8\lambda^2\mu^2 = \pm Pr^2 \;\ldots\ldots\ldots\ldots\ldots\; \text{(A)},$$
μ being odd, λ odd or even, and λ or μ being taken negatively as well as positively.

The form of P is $8a\pm 1$.

If P is of the form $8a-1$, we must have a negative result, so that, absolutely,
$$2\lambda^2+\mu^2 < 2\lambda\mu\,\{2+\sqrt{2}\},$$
and λ or μ must be taken with a negative sign.

(II.) $a = k^2,\; b = l^2,\; k^2-l^2 = \pm m^2,\; k^2+l^2 = Pn^2.$

Writing $k = \lambda^2+\mu^2$, $l = 2\lambda\mu$, or *vice versâ*, we get

$$(\lambda^2+\mu^2)^2+4\lambda^2\mu^2 = \frac{(\lambda+\mu)^4+(\lambda-\mu)^4}{2} = Pn^2 \;\ldots\ldots\ldots\; \text{(B)},$$

λ, μ being of different species.

The form of P is $8a+1$.

(III.) $a = k^2,\; b = Pl^2,\; k^4-P^2l^4 = \pm m^2.$

(1) $k^4-P^2l^4 = m^2$. This is equivalent to the original system; and if the derived equation belongs to the same case, we should now have $b = Pl'^2$, $l' < l$. Hence, by Fermat's principle, we must arrive at one of the other cases.

(2) $P^2l^4-k^4 = m^2$. We write $Pl^2 = p^2+q^2$, $k^2 = 2pq$, and $p = r^2$, $q = 2s^2$. Then
$$r^4+4s^4 = Pl^2.$$

But we may now write
$$(r-s)^2+s^2 = Pt^2,$$
$$(r+s)^2+s^2 = \tau^2;$$
and we have two sub-cases; namely,
$$r+s = \lambda^2-\mu^2,\ s = 2\lambda\mu,$$
giving
$$(\lambda^2-\mu^2-4\lambda\mu)^2+4\lambda^2\mu^2 = Pt^2 \text{ .... (C)};$$
and
$$r+s = 2\lambda\mu,\ s = \lambda^2-\mu^2,$$
giving
$$4(\lambda^2-\mu^2-\lambda\mu)^2+(\lambda^2-\mu^2)^2 = Pt^2 \text{ .... (D)};$$
where (C) gives P of the form $8a+1$, and (D) gives it of the form $8a+5$, and includes all the admissible prime values of P of that form. We take λ, μ of opposite species, and positively or negatively.

The cases are now exhausted, and it appears, as shown by Prof. Genocchi, that P cannot be a prime of the form $8a+3$.*

3. The formulæ arrived at are somewhat rich in primes for low values of λ and μ. As to the forms $8a+1$, the results overlap, and the same values, *e.g.* 41, may satisfy (A), (B), and (C).

I give a few results of the different formulæ—

μ,	λ,	P	
1,	−1,	7	........................(A),
1,	1,	41	
1,	−2,	31	
1,	−3,	23	
3,	−1,	71	
9,	−2,	47	
1,	2,	41	........................(B),
3,	2,	313	
1,	4,	353	
−2,	1,	41	........................(C),
2,	1,	137	
0,	1,	5	........................(D),
1,	2,	13	
1,	−2,	109	
2,	3,	29	
5,	−2,	37	
12,	−1,	53	
8,	−5,	61	

Numerous high primes appear in all the series for low values of λ, μ.

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* "Note Analitiche," p. 103, or Tortolini, vi., p. 315.

4. Having regard to the double of a prime, say $2P$, we may write $2b$ for b in (a), a being odd and b odd or even. We then have the condition $$ab\,(a^2-4b^2) = \pm P\,.\,V^2.$$

The cases to be considered are

I. $$a = k^2,\ b = l^2,\ k^2+2l^2 = m^2,\ k^2-2l^2 = \pm Pn^2.$$

Here l cannot be odd, and writing
$$k = \lambda^2-2\mu^2,\ l = 2\lambda\mu,$$
we get $$(\lambda^2-2\mu^2)^2-8\lambda^2\mu^2 = \pm Pn^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{A}'),$$
λ being odd, and μ odd or even.

The forms of P are $8\alpha \pm 1$.

II. $$a = k^2,\ b = l^2,\ k^2-2l^2 = m^2,\ k^2+2l^2 = Pn^2.$$

Since l cannot be odd, we write $k = \lambda^2+2\mu^2,\ l = 2\lambda\mu$,
and then $$(\lambda^2+2\mu^2)^2+8\lambda^2\mu^2 = Pn^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B}').$$

The form of P is $8\alpha+1$.

III. $$a = k^2,\ b = l^2,\ 2l^2-k^2 = m^2,\ k^2+2l^2 = Pn^2.$$

Writing $$l = \lambda^2+\mu^2,\ k = \lambda^2-\mu^2+2\lambda\mu,$$
we get $$(\lambda^2-\mu^2+2\lambda\mu)^2+2\,(\lambda^2+\mu^2)^2 = Pn^2 \quad\ldots\ldots\ldots\ldots(\text{C}'),$$
λ, μ being of opposite species, and being taken negatively and positively.

The form of P is $8\alpha+3$.

IV. $$a = k^2,\ b = Pl^2,\ k^4-4P^2l^4 = m^2;$$
and therefore $$k^2+2Pl^2 = r^2,\ k^2-2Pl^2 = s^2,$$
the original system as to form; and, therefore, by Fermat's principle, we must arrive at one of the other cases, as in the analogous case of primes.

V. $$a = k^2,\ \ b = Pl^2,\ 4P^2l^2-k^4 = m^2$$
is impossible, k being odd.

VI. $$a = Pk^2,\ \ b = l^2,\ \ P^2k^4-4l^4 = m^2.$$

Here l must be even, and we may write
$$Pk^2 = p^2+q^2,\ l^2 = pq,\ p = \lambda^2,\ q = \mu^2,$$
giving $$\lambda^4+\mu^4 = Pk^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{D}'),$$
λ, μ being of opposite species.

The form of P is $8\alpha+1$.

VII. $$a = Pk^2,\ b = l^2,\ 4l^4-P^2k^4 = m^2$$
is an impossible case.

From the foregoing results, it appears that P cannot be of the form

$8a+5$*, but may be of the form $8a+3$, and these primes are included in one formula (C′) exclusively.

Hence $x^3+P^2y^4$ in integers cannot be a square if P is a prime of form $8a+5$.

I give a few results of the different formulæ—

λ,	μ,	P	
1,	1,	7	……………………(A′),
1,	2,	17	
3,	1,	23	
7,	10,	31	
1,	1,	17	……………………(B′),
1,	2,	113	
3,	1,	193	
1,	0,	3	……………………(C′),
1,	−2,	11	
1,	6,	3	
3,	−2,	43	
3,	4,	19	
1,	2,	17	……………………(D′).
1,	4,	257	
3,	2,	97	

5. When P is a given prime of one of the possible forms, it is not easy to determine by a definite process whether it is admissible or not. If we can transform the system into another of obvious solution, or involving inconsistency of form, the question may be answered. Solutions obtained by forming tables of "congruous quantities," the differences of fourth powers, &c., are more or less accidental, and the formulæ I have given are, of course, open to the same objection.

We may, however, employ these formulæ to determine the inadmissibility of certain primes, and for this purpose the forms of the results must be examined more closely.

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* For the case of $P=5$, see "Note Analitiche," p. 87, or Tortolini, vi., p. 299.

I have found, however, since the above was written, that M. Genocchi has published the result as to primes of the form $8m+5$ ("Comptes Rendus," 1874, lxxviii., pp. 433–6), "Sur l'impossibilité de quelques inégalités doubles." The author has considered also the product of two primes. Thus the product of two primes of form $8m+3$, and the double of the product of two primes of form $8m+5$, are alike inadmissible. The communication contains only the enunciations of results, and I do not know whether the proofs have been elsewhere given.

Let P be a prime of the form $8a+1$, the equation (A) gives Pr^2 of the forms

$$(8a-1)^2-8(2\beta+1)^2 = 16t+9, \quad \lambda \text{ odd},$$
$$(8a+1)^2-8(2\beta)^2 = 16t+1, \quad \lambda \text{ even};$$

for $2(\lambda+\mu)^2-\mu^2$ is essentially positive under the condition

$$[2(\lambda+\mu)^2-\mu^2]^2 > 8\lambda^2\mu^2.$$

Let $P=(8a+s)^2-8k^2$, where $s=\pm 1$ or ± 3, and

$$r^2=(p^2+2q^2)^2-8p^2q^2=(p^2-2q^2)^2.$$

Then $Pr^2=[(8a+s)(p^2+2q^2)+8kpq]^2-8[k(p^2+2q^2)+(8a+s)pq]^2$.

If k is odd, we have the forms

$$(8a'+3s)^2-8(2\beta')^2, \quad q \text{ odd}, \quad \lambda \text{ even},$$
$$(8a'+s)^2-8(2\beta'+1)^2, \quad q \text{ even}, \quad \lambda \text{ odd}.$$

The consistent values of s are $+3$ and -1, for

$$(8a+s)(p^2+2q^2)+8kpq$$

is essentially positive for a positive.

If k is even, the forms are

$$(8a'+3s)^2-8(2\beta'+1)^2, \quad q \text{ odd}, \quad \lambda \text{ odd},$$
$$(8a'+s)^2-8(2\beta')^2, \quad q \text{ even}, \quad \lambda \text{ even}.$$

The consistent values of s are -3 and $+1$.

And we have the following inconsistent and inadmissible forms of P:—

$$(8a-3)^2-8(2\beta+1)^2 \equiv 1 \pmod{16},$$
$$(8a+1)^2-8(2\beta+1)^2 \equiv 9 \pmod{16},$$
$$(8a+3)^2-8(2\beta)^2 \equiv 9 \pmod{16},$$
$$(8a-1)^2-8(2\beta)^2 \equiv 1 \pmod{16}.$$

Next, as to the formulæ (B) and (C):

The forms for (B) are

$$(8a-3)^2+16(2\beta+1)^2, \quad \mu=2(2\beta+1), \quad \lambda \text{ odd},$$
$$(8a+1)^2+16(2\beta)^2, \quad \mu=4\beta, \quad \lambda \text{ odd}.$$

In (C) the forms are

$$(8a\pm 3)^2+16(2\beta+1)^2, \quad \mu=2(2\beta+1), \quad \lambda \text{ odd},$$
$$(8a\pm 1)^2+16(2\beta)^2, \quad \mu=4\beta, \quad \lambda \text{ odd}.$$

Now, if $P=(8a+s)^2+16k^2$, Pn^2 or Pt^2 may be written, if n^2 or t^2 $=(p^2+q^2)^2=(p^2+4q'^2)^2$,

$$\{(8a+s)(p^2-4q'^2)+16kpq'\}^2+16\{k(p^2-4q'^2)+(8a+s)pq'\}^2.$$

And if k be odd, the forms are

$$\left.\begin{array}{lll}(8\alpha'\pm 3s)^2+16\,(2\beta')^2, & q' \text{ odd}, & \mu' \text{ even} \\ (8\alpha'\pm s)^2\ +16\,(2\beta'+1)^2, & q' \text{ even}, & \mu' \text{ odd}\end{array}\right\}\ldots\ldots\ldots\ldots(\alpha),$$

where $\qquad 2\mu'=\mu.$

The consistent values of s are ± 3 for (B) and (C).

If k be even, the forms are

$$\left.\begin{array}{lll}(8\alpha'\pm 3s)^2+16\,(2\beta'+1)^2, & q' \text{ odd}, & \mu' \text{ odd} \\ (8\alpha'\pm s)^2\ +16\,(2\beta')^2, & q' \text{ even}, & \mu' \text{ even}\end{array}\right\}\ldots\ldots\ldots(\alpha').$$

The consistent forms of s are ± 1.

Hence, we have the inadmissible forms of P,

$$(8\alpha\pm 1)^2+16\,(2\beta+1)^2,$$
$$(8\alpha\pm 3)^2+16\,(2\beta)^2.$$

Putting these results together, we get the following criteria for inadmissible forms of P:—

$$\left.\begin{array}{rl} P &= (8\alpha\ \pm 1)^2+16\,(2\beta+1)^2 \\ &= (8\alpha'-3)^2-\ 8\,(2\beta'+1)^2 \\ \text{or} &= (8\alpha''-1)^2-\ 8\,(2\beta'')^2\end{array}\right\}\equiv 1 \text{ (mod. 16)},$$

$$\left.\begin{array}{rl} P &= (8\ \ \pm 3)^2+16\,(2\beta)^2 \\ &= (8\alpha'+3)^2-\ 8\,(2\beta')^2 \\ \text{or} &= (8\alpha''+1)^2-\ 8\,(2\beta'+1)^2\end{array}\right\}\equiv 9 \text{ (mod. 16)}.$$

6. The forms of $p^2-2q^2=1$ are, however,

$$(8\alpha+1)^2-8\,(2\beta)^2,$$

or
$$(8\alpha+3)^2-8\,(2\beta+1)^2.$$

Hence any positive number P^2-8Q^2 may be put into two forms

$$(8\alpha+s)^2-8\beta^2,$$
$$(8\alpha'+3s)-8\beta'^2,$$

where $s=\pm 1$ and β, β' are of different species. We may, therefore, omit one of the corresponding alternative forms in the above criteria.

But, further, if a number of the form $(8\alpha\pm 1)^2+16\,(2\beta+1)^2$, when the squares are relatively prime, is transformable into P^2-8Q^2, we cannot have the form $\qquad (8\alpha''+1)^2-8\,(2\beta'')^2$.

For, then, $(8\alpha\pm 1)^2+16\,(2\beta+1)^2=(8\alpha''+1)^2-32\beta''^2,$

$$64\,(\alpha''^2-\alpha^2)+16\,(\alpha''\mp\alpha)=16\,(2\beta+1)^2+32\beta''^2,$$
$$(\alpha''\mp\alpha)\,[4\,(\alpha''\pm\alpha)+1]=(2\beta+1)^2+2\beta''^2\,;$$

and $\alpha''\mp\alpha$ being odd, $4\,(\alpha''\pm\alpha)+1$ is of the form $8n-3$, and $\alpha''\mp\alpha$ must be of the form $8m-1$ or $8m-3$. Now, a prime factor of these forms must enter squared, if at all; and then $\alpha''\mp\alpha$ and $4\,(\alpha''\pm\alpha)+1$, and consequently $8\alpha \pm 1$ and $8\alpha''+1$, must respectively contain such a factor in common with $2\beta+1$ and β'', contrary to supposition.

Similarly, it may be shown that a number $(8\alpha \pm 3)^2+16\,(2\beta)^2$ transformed into P^2-8Q^2 cannot be of the form $(8\alpha'-3)^2-8\,(2\beta')^2$, P and Q being relatively prime. And we may add, that

$$(8\alpha \pm 1)^2+16\,(2\beta)^2$$

cannot be transformed into $(8\alpha''-1)^2-8\,(2\beta'')^2$, or $(8\alpha \pm 3)^2+16\,(2\beta+1)^2$ into $(8\alpha'+3)^2-8\,(2\beta')^2$.

We find, then, generally, that primes of the forms

$$(8\alpha \pm 1)^2+16\,(2\beta+1)^2,$$

and

$$(8\alpha \pm 3)^2+16\,(2\beta)^2,$$

are inadmissible.

In other words, if P be such a prime,

$$x^4+4P^2y^4 = u^2$$

is impossible in whole numbers.

7. All these excluded primes are such that the doubles of them are not decomposable into sums of two squares by continued fractions; *i.e.*, cannot satisfy $x^2-2Py^2 = -1$ ....(b), x and y being integers.

In fact, we should have [see results (a), (a')]

$$Py^2 = p^2+(p-1)^2,$$

which is inconsistent with the above forms, since the squared numbers differ by ± 3 in form relative to the modulus 8.

The consistent class of primes does, however, comprise numbers which fail to satisfy the equation (b).

As to primes of the form $8n-1$ or $8n+5$, I do not find any corresponding criteria of exclusion. Nor, when we consider P as the double of a prime, have I detected any inconsistency depending on the modulus 8.

I do not know that any odd prime >1, not excluded by the criteria, or of the form $8n+3$, or that any double, other than 2, of an odd prime, not of the form $2\,(8n+5)$, has been shown to be inadmissible.

In the following list of primes of the form $8m+1$, those marked with an asterisk are excluded :—

17*	113	257	401*
41	137	281*	409
73*	193*	313	433*
89*	233*	337	449*
97*	241*	353	457

8. Although the present subject does not require any development of the remark at the commencement of the last article, I may as well mention here the more general form of the result. The conclusion stated is founded on the fact that a prime can be represented as the sum of two integer squares only in one way.

But we may consider P as compounded of prime factors of the form $8m+1$, so that $P=A^2+16B^2=C^2-8D^2$, where A and B, C and D, are relative primes.

I. It follows from §6, that all the decompositions $\alpha^2+\beta^2$ are of the same form; namely,

$$(8\alpha+s)^2+16\,(2\beta+1)^2,\quad s=\pm 1 \text{ or } \pm 3,$$

or

$$(8\alpha+s)^2+16\,(2\beta).$$

II. And, further, that if P is of the form $(8\alpha\pm 1)^2+16\,(2\beta+1)^2$, or $(8\alpha+3)^2+16\,(2\beta^2)$, the equation $x^2-2Py^2=-1$ cannot hold in integers, and the method of continued fractions fails to give the decompositions of $2P$ into the sum of two square integers.

As to (I.), for example; suppose that P can be put into the form $(8\alpha\pm 1)^2+16\,(2\beta)^2$, and also in the form $(8\alpha\pm 1)^2+16\,(2\beta+1)^2$. Then P can be put into the forms $(8\alpha-1)^2-8\,(2\beta)^2$ and $(8\alpha+1)^2-8\,(2\beta)^2$, which forms are mutually exclusive, by §6.

The conclusion evidently does not hold for numbers having prime factors of the form $8m+5$; *e.g.*,

$$145=5\,.\,29=1^2+16\,.\,3^2=9^2+16\,.\,2^2.$$

The result (II.) follows as in the case of P prime.

December 11*th*, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The following Members were elected:—

Mr. W. Burnside, M.A., Fellow of Pembroke College, Cambridge; Mr. J. R. Harris, M.A., Fellow of Clare College, Cambridge; Dr. W. Jack, formerly Fellow of St. Peter's College, Cambridge, Professor of Mathematics in the University of Glasgow; Mr. W. J. Curran Sharp, M.A., Ex-scholar of Trinity College, Dublin; Prof. W. Woolsey Johnson, St. John's College, Annapolis, Maryland, U.S.

The Auditor (Mr. R. F. Scott, M.A.) stated that he had been to the Bank of England, and found that the Treasurer's statement as to the monies therein deposited, was true, and that he had examined the Accounts and found them correct.

On the proposal of the Chairman, a vote of thanks was passed to the Auditor.

The following communications were made :—

"Note on a Method of obtaining the q-formula for the Sine-amplitude in Elliptic Functions :" Mr. J. W. L. Glaisher, F.R.S.

"Note on a Numerical Theorem connected with the Cubical Division of Space :" the President.

"Notes on Curvature :" Mr. J. J. Walker, M.A.

"A Property of a Linkage :" Mr. A. B. Kempe, B.A.

The following presents were made to the Society :—

"Proceedings of the Royal Society," Vol. xxix., No. 198.

"Educational Times," December, 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Août, 1879.

"American Journal of Mathematics, Pure and Applied," Vol. ii., No. 3, September, 1879 ; Baltimore.

Note on a Method of obtaining the q-formula for the Sine-amplitude in Elliptic Functions. *By* J. W. L. Glaisher, M.A., F.R.S.

[*Read December* 11*th*, 1879.]

The present Note contains a very short method of deducing the q-series for sn u from the q-products for sn u, cn u, dn u.

1. The products are

$$\operatorname{sn} u = \frac{1}{k^{\frac{1}{2}}} \,.\, 2q^{\frac{1}{4}} \sin x \, \Pi \, \frac{1-2q^{2n}\cos 2x + q^{4n}}{1-2q^{2n-1}\cos 2x + q^{4n-2}} \;\ldots\ldots\ldots\ldots \text{(i.)},$$

$$\operatorname{cn} u = \frac{k'^{\frac{1}{2}}}{k^{\frac{1}{2}}} \,.\, 2q^{\frac{1}{4}} \cos x \, \Pi \, \frac{1+2q^{2n}\cos 2x + q^{4n}}{1-2q^{2n-1}\cos 2x + q^{4n-2}} \;\ldots\ldots\ldots\ldots \text{(ii.)},$$

$$\operatorname{dn} u = k'^{\frac{1}{2}} \qquad\qquad . \, \Pi \, \frac{1+2q^{2n-1}\cos 2x + q^{4n-2}}{1-2q^{2n-1}\cos 2x + q^{4n-2}} \;\ldots\ldots\ldots \text{(iii.)},$$

where u denotes $\dfrac{2Kx}{\pi}$, and the product symbol Π refers to the letter n, which is to have all positive values from $n = 1$ to $n = \infty$.

Taking the logarithm of (iii.), and differentiating, we have, as in the *Fundamenta Nova*, pp. 99, 100,

$$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{\pi}{2k^2K} \left\{ \frac{8q}{1-q^2} \sin 2x + \frac{8q^3}{1-q^6} \sin 6x + \&c. \right\} ;$$

and multiplying (i.) and (ii.), and dividing by (iii.),

$$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{1}{k} \,.\, 2q^{\frac{1}{2}} \sin 2x \, \Pi \, \frac{1-2q^{4n}\cos 4x + q^{8n}}{1-2q^{4n-2}\cos 4x + q^{8n-4}} \;\ldots\ldots\text{(iv.)},$$

so that $$2q^{\frac{1}{2}} \sin 2x \, \Pi \, \frac{1-2q^{4n}\cos 4x + q^{8n}}{1-2q^{4n-2}\cos 4x + q^{8n-4}}$$

$$= \frac{\pi}{2kK} \left\{ \frac{8q}{1-q^2} \sin 2x + \frac{8q^3}{1-q^6} \sin 6x + \&\text{c.} \right\}.$$

In this equation we observe that, if $\frac{1}{2}x$ be substituted for x, and $q^{\frac{1}{2}}$ for q, the left-hand member becomes equal to $k^{\frac{1}{2}} \operatorname{sn} u$, from (i.), so that

$$k^{\frac{1}{2}} \operatorname{sn} u = M \left\{ \frac{8q^{\frac{1}{2}}}{1-q} \sin x + \frac{8q^{\frac{3}{2}}}{1-q^3} \sin 3x + \&\text{c.} \right\},$$

where M is what $\dfrac{\pi}{2kK}$ becomes when q is replaced by $q^{\frac{1}{2}}$.

Now, by putting $x = 0$ in (iv.) and (i.),

$$\frac{2kK}{\pi} = 4q^{\frac{1}{2}} \, \Pi \left(\frac{1-q^{4n}}{1-q^{4n-2}} \right)^2,$$

$$\frac{2k^{\frac{1}{2}}K}{\pi} = 2q^{\frac{1}{4}} \, \Pi \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right)^2;$$

the change of q into $q^{\frac{1}{2}}$, therefore, converts $\dfrac{2kK}{\pi}$ into $\dfrac{4k^{\frac{1}{2}}K}{\pi}$, so that

$$M = \frac{\pi}{4k^{\frac{1}{2}}K};$$

and therefore $$\operatorname{sn} u = \frac{\pi}{2kK} \left\{ \frac{4q^{\frac{1}{2}}}{1-q} \sin x + \frac{4q^{\frac{3}{2}}}{1-q^3} \sin 3x + \&\text{c.} \right\}.$$

2. By this method we arrive almost immediately at the q-series for $\operatorname{sn} u$; and it may be observed that this is the only one of the q-series of which Jacobi makes any important use in the subsequent portion of the *Fundamenta Nova.* This use is in connexion with the function Z (u), which is obtained by squaring the series for $\operatorname{sn} u$ and integrating.

In the *Fundamenta Nova* the q-products arise as limiting cases of formulæ in the transformation theory; these formulæ afford products, not only for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, but also for

$$\left(\frac{1-\operatorname{sn} u}{1+\operatorname{sn} u} \right)^{\frac{1}{2}} \text{ and } \left(\frac{1-k \operatorname{sn} u}{1+k \operatorname{sn} u} \right)^{\frac{1}{2}};$$

and it is from the latter of these expressions, which, differentiated logarithmically, gives $k \operatorname{sn}(K-u)$, that Jacobi deduces the series for $\operatorname{sn} u$. But if the q-products are derived otherwise than from the transformation theory (as, *e.g.*, from the Abel double products), so that it is convenient to start with the products for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, the deduction of the series for $\operatorname{sn} u$ requires an imaginary transformation of either the product or the series; *i.e.*, we may either form the q-product for $\frac{\operatorname{cn}(\frac{1}{2}u+\frac{1}{2}iK')\operatorname{dn}(\frac{1}{2}u+\frac{1}{2}iK')}{\operatorname{sn}(\frac{1}{2}u+\frac{1}{2}iK')}$, which, when differentiated logarithmically, gives $-k\operatorname{sn}u$;* or, we may form the q-product for $\frac{\operatorname{cn}\frac{1}{2}u\operatorname{dn}\frac{1}{2}u}{\operatorname{sn}\frac{1}{2}u}$, which, when differentiated logarithmically, gives $-\frac{1}{\operatorname{sn}u}$; and then transform the series by the substitution of $u+iK'$ for u. This latter method is followed by Durège (*Theorie der elliptischen Functionen*, 1861, and second edition, 1868).

In the process given above, the substitution of $u+iK'$ for u, either in the products or the series, is avoided, and the series for $\operatorname{sn}u$ is obtained very readily from the q-formulæ; but the method is only applicable in the case of the $\operatorname{sn}u$, and does not give the series for $\operatorname{cn}u$, $\operatorname{dn}u$, $\frac{1}{\operatorname{sn}u}$, &c.

January 8th, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Prof. W. Snow Burnside, M.A., Trinity College, Dublin, was elected a Member.

Prof. Cayley read a paper, "A Theorem in Spherical Trigonometry," and made some remarks upon a posthumous paper of Prof. Clifford's, "On the Theory of Distances."

The Chairman supplemented his communication of December 11th by a few further remarks.

Mr. J. Hammond, M.A., gave an Expression for the Complementary Function in Fractional Differentiation; upon which a discussion arose, in which Prof. Cayley, the Chairman, Mr. Roberts, Mr. Glaisher, and Mr. A. Freeman took part.

The following presents were received:—

"Zur Theorie der Elliptischen Modulfunctionen," von F. Klein (aus den Sitzungsberichten der Akad. der Wissenschaften zu München, vom 6 Dezember, 1879).

"The Melbourne University Calendar for the Academic years 1878—1880," Melbourne: from the Chancellor of the University.

"Factor Table for the Fourth Million, containing the least factor of every number not divisible by 2, 3, or 5, between 3,000,000 and 4,000,000," by J. Glaisher, F.R.S., 1879: from the author.

"Crelle's Journal," 88 Band, 3^{es} Heft, Berlin, 1879; 4^{es} Heft, Berlin, 1880.

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* This method is employed in "A Chapter in Elliptic Functions" (*Quarterly Journal*, T. xvii., pp. 50—65, 1880).

"Beiblätter zu den Annalen der Physik und Chemie," Band iii., Stück 12, Leipzig, 1879; Band iv., Stück 1, Leipzig, 1880.

"Educational Times," January, 1880.

"Teorica delle Forze Newtoniane e sue applicazioni all' elettrostattica e al magnetismo," del Prof. Enrico Betti; Pisa, 1879.

"Journal of Institute of Actuaries," No. cxvii., Vol. xxii., Part i., October, 1879.

"Proceedings of the Royal Society," Vol. xxix., No. 199.

"Monatsbericht" (Sept. and Oct.), Nov., 1879.

"Atti della R. Accademia dei Lincei—Transunti," Vol. iv., Fasc. 1°, Dic. 1879; Roma, 1880.

"Bulletin de la Société Mathématique de France," Tome vii., No. 6; Paris, 1879.

"Archives Néerlandaises des Sciences Exactes et Naturelles," publiées par la Société Hollandaise des Sciences à Harlem, Tome xiv. (3^{me}, 4^{me}, et 5^{me} Livr); La Haye, 1879.

"The Physical Society of London—Proceedings," Vol. iii., Part 3, Nov. 1879—Feb. 1880.

"Tidsskrift for Mathematik," udgivet af H. G. Zeuthen, Fjerde Række, Tredie Aargang, Fφrste . . . Sjette Hefte; Kjφbenhavn, 1879.

A Theorem in Spherical Trigonometry. By Prof. Cayley.

[*Read January* 8*th*, 1880.]

In a spherical triangle, where a, b, c are the sides, and A, B, C the opposite angles, we have

$$-\tan\tfrac{1}{2}c\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin(A-B)=\tan\tfrac{1}{2}b\sin A-\tan\tfrac{1}{2}a\sin B,$$

$$\tan\tfrac{1}{2}c\,\{1-\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\}=\tan\tfrac{1}{2}b\cos A+\tan\tfrac{1}{2}a\cos B;$$

which are both included in the form

$$\tan\tfrac{1}{2}a\,(\cos B-i\sin B)=\frac{\tan\frac{1}{2}c-\tan\frac{1}{2}b\,(\cos A+i\sin A)}{1+\tan\frac{1}{2}c\tan\frac{1}{2}b\,(\cos A+i\sin A)}.$$

For the first of the two identities: from

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C},$$

$$\cos b=\frac{\cos B+\cos A\cos C}{\sin A\sin C},$$

we deduce

$$\cos a - \cos b = \frac{1}{\sin C}\left(\frac{\cos A}{\sin B} - \frac{\cos B}{\sin A}\right) + \frac{\cos C}{\sin C}\left(\frac{\cos B}{\sin B} - \frac{\cos A}{\sin A}\right)$$

$$= \frac{1}{\sin C} \cdot \frac{\frac{1}{2}(\sin 2A - \sin 2B)}{\sin A \sin B} + \frac{\cos C \,.\, \sin (A-B)}{\sin C \sin A \sin B}$$

$$= \frac{\sin (A-B)}{\sin C \sin A \sin B}\{\cos (A+B) + \cos C\}$$

$$= \frac{\sin (A-B)}{\sin C}(\cos\ -1\ );$$

that is,
$$-\sin (A-B) = \frac{\sin C}{1-\cos c}(\cos a - \cos b),$$

$$= \frac{\sin C}{\sin c}\ \frac{\sin c}{1-\cos c}\ (\cos a - \cos b);$$

or, what is the same thing,

$$-\tan \tfrac{1}{2}c \sin (A-B) = \frac{\sin C}{\sin c}(\cos a - \cos b).$$

Here $\cos a - \cos b$ is $= (1+\cos a) - (1+\cos b)$, and for $\dfrac{\sin C}{\sin c}$ substituting successively $\dfrac{\sin A}{\sin a}$ and $\dfrac{\sin B}{\sin b}$, the right-hand side is

$$= \frac{1+\cos a}{\sin a}\sin A - \frac{1+\cos b}{\sin b}\sin B,$$

$$= \cot \tfrac{1}{2}a \sin A - \cot \tfrac{1}{2}b \sin B;$$

whence, multiplying each side by $\tan \frac{1}{2}a \tan \frac{1}{2}b$, we have the relation in question.

For the second identity,

$$\tan \tfrac{1}{2}c\,\{1 - \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \cos (A-B)\} = \tan \tfrac{1}{2}b \cos A + \tan \tfrac{1}{2}a \cos B;$$

if on the right-hand side we substitute for $\cos A$, $\cos B$ their values $\dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$ and $\dfrac{\cos b - \cos a \cos c}{\sin a \sin c}$, the right-hand side becomes

$$\frac{1}{\sin c}\left\{\frac{\cos a - \cos b \cos c}{1+\cos b} + \frac{\cos b - \cos a \cos c}{1+\cos a}\right\};$$

whence, multiplying the whole equation by $\sin c\,(1+\cos a)(1+\cos b)$, it becomes

$$(1-\cos c)\{(1+\cos a)(1+\cos b) - \sin a \sin b \cos (A-B)\}$$
$$= (1+\cos a)(\cos a - \cos b \cos c) + (1+\cos b)(\cos b - \cos c \cos a).$$

We have here $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \frac{(\cos a - \cos b \cos c)(\cos b - \cos c \cos a) + \square}{\sin^2 c \sin a \sin b},$$

by substituting for $\cos A$, $\cos B$ their foregoing values, and for $\sin A$, $\sin B$ their values $\frac{\sqrt{\square}}{\sin b \sin c}$, $\frac{\sqrt{\square}}{\sin a \sin c}$, where

$$\square = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

The numerator is

$$\begin{aligned} &\cos a \cos b - \cos c (\cos^2 a + \cos^2 b) + \cos a \cos b \cos^2 c \\ &+ 1 - \cos^2 c - (\cos^2 a + \cos^2 b) + \cos a \cos b \,.\, 2 \cos c; \end{aligned}$$

viz., this is

$$= \cos a \cos b (1 + \cos c)^2 - (\cos^2 a + \cos^2 b)(1 + \cos c) + 1 - \cos^2 c,$$

having the factor $1 + \cos c$, which is also a factor of $\sin^2 c$, $= 1 - \cos^2 c$, in the denominator. We have, therefore,

$$\cos(A-B) = \frac{\cos a \cos b (1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c}{(1 - \cos c) \sin a \sin b};$$

and the equation thus is

$$\begin{aligned} &(1 - \cos c)(1 + \cos a)(1 + \cos b) \\ &\qquad - \{\cos a \cos b (1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c\} \\ &= (1 + \cos a)(\cos a - \cos b \cos c) + (1 + \cos b)(\cos b - \cos c \cos a), \end{aligned}$$

where each side is in fact

$$= \cos a + \cos^2 a + \cos b + \cos^2 b - \cos c (\cos a + \cos b) - 2 \cos a \cos b \cos c;$$

and the second identity is thus proved.

February 12*th*, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. D. Edwardes was elected a Member, and subsequently admitted into the Society.

The following communications were made to the Society :—

"Geometrical Notes" (3) : by Prof. H. J. S. Smith, F.R.S.

"On the Reflection of Vibrations at the confines of two Media between which the Transition is Gradual," and "On the Stability, or Instability, of certain Fluid Motions" : by Lord Rayleigh, F.R.S.

"The Calculus of Equivalent Statements" (Fourth Note) : by Mr. H. McColl, B.A.

The following presents were received:—

"Nogle Egenskaber ved Kurver af fjerde Orden med to Dobbeltpunkter;" af H. G. Zeuthen.

"Reprint of Mathematics from Educational Times," Vol. xxxii.

"Beiblätter zu den Annalen der Physik und Chemie," Band iv., Stück 2, 1880.

"Proceedings of Royal Society," Vol. xxx., No. 200.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^me^ Série, Tome iii., Sept. 1879; Paris, 1879.

"Educational Times," February, 1880.

"Jahrbuch über die Fortschritte der Mathematik," neunter Band, Jahrgang 1877, Heft 3; Berlin, 1880.

"Sitzungsberichte der physikalisch-medicinischen Societät zu Erlangen," 5 Heft, Nov. 1872—Aug. 1873; 6 Heft, Nov. 1873—Aug. 1874. . . . . 11 Heft, Nov. 1878—Aug. 1879. (7 Vols.)

"Atti della R. Accademia dei Lincei—Transunti," Vol. iv., Fasc. 2, Gennaio, 1880; Roma, 1880.

"Bibliographie générale de l'Astronomie, ou Catalogue méthodique des ouvrages, des mémoires et des observations astronomiques, publiés depuis l'origine de l'imprimerie jusqu'en 1880:" from J. C. Houzeau and A. Lancaster.

"Ueber Relationen zwischen Klassenzahlen binärer quadratischer Formen von negativer Determinante," von J. Gierster in Bamberg.

On Reflection of Vibrations at the Confines of two Media between which the Transition is Gradual. *By* Lord RAYLEIGH, F.R.S., Professor of Experimental Physics in the University of Cambridge.

[*Read February* 12*th*, 1880.]

Many physicists, of whom may be especially mentioned Young, Fresnel, Poisson, Green, and Cauchy, have investigated the reflection of light or sound at the surface of separation of two uniform media, of different mechanical properties. The transition from one medium to the other being treated as abrupt, the problem is of no great difficulty for the case of plane waves incident upon a plane surface of separation. It is of some interest to inquire what modifications would be caused by the substitution of a gradual for an abrupt transition, and the principal object of this paper is to give the details of one particular case, which admits of pretty simple treatment.

It is evident, from the first, that the transition may be supposed to be so gradual that no sensible reflection would ensue. No one would expect a ray of light to undergo reflection in passing through the earth's atmosphere as a consequence of the gradual change of density with elevation. At first sight, indeed, the case of so-called total reflection may appear to be an exception, as it is independent of the suddenness of transition; but this only shows that the phenomenon is inaccurately described by its usual title. It is, in strictness, a particular case of refraction, rather than of reflection, and must be so considered in theoretical work, although, no doubt, the name of total reflection will be retained whenever, as in constructing optical instruments, we have to deal with effects rather than with causes.

Admitting, then, that reflection proper is due to suddenness of transition, we have still to inquire what degree of suddenness is requisite. It is not difficult to see that the quantity with which the thickness of the transitional layer comes into comparison is the wave-length of the vibration; so that, when the thickness is a large multiple of the wave-length, there is little reflection, but when, on the other hand, the wave-length is a large multiple of the thickness, the reflection is sensibly as copious as if the transition were absolutely abrupt. There is thus a considerable distinction, in practice, between the cases of sound and of light, the wave-length of the one being some million times greater than that of the other. When sound is reflected in air from ordinary solids or liquids, the transition between the media may be treated as abrupt with abundant accuracy; but it is not so certain that a similar treatment is adequate to the case of light. It is probable, however, that in the case of light passing from fluids to solids, or from one fluid to a second which does not mix with it—*e.g.*, from bisulphide of carbon to water—the phenomena are not materially influenced by insufficient suddenness of transition. On the other hand, when the two fluids are miscible, it would not be easy to complete an experiment before the abruptness of transition is so far broken down by diffusion that the thickness of the transitional layer amounts to a multiple of $\frac{1}{40000}$ of an inch.

The problem of gradual transition includes, of course, that of a variable medium. The particular case which I have taken is that of a stretched string, whose longitudinal density varies as the inverse square of the abscissa (measured along its length), vibrating transversely. The same analysis is strictly applicable to other cases of vibrations, which I need not stop to specify, and the results will illustrate the general character of the phenomena to be expected in all cases. If y, denoting the transverse displacement, be proportional to e^{ipt}, the equation which it must satisfy as a function of of x, is

$$\frac{d^2y}{dx^2}+n^2x^{-2}y=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(1),$$

where n^2 is some positive constant, of the nature of an abstract number.*

The solution of (1) is $y=Ax^{\frac{1}{2}+im}+Bx^{\frac{1}{2}-im}$ $\dots\dots\dots\dots\dots\dots\dots\dots(2)$,

where $$m^2=n^2-\tfrac{1}{4} \quad \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

If m be real, that is, if $n>\frac{1}{2}$, we may obtain, by supposing $A=0$, as a final solution in real quantities,

$$y=Cx^{\frac{1}{2}}\cos(pt-m\log x+\epsilon) \quad \dots\dots\dots\dots\dots\dots(4),$$

which represents a positive progressive wave, in many respects similar to those propagated in uniform media.

Let us now suppose that, to the left of the point $x=x_1$, the variable medium is replaced by one of uniform constitution, such that there is no discontinuity of density at the point of transition; and let us inquire, what reflection a positive progressive wave in the uniform medium will undergo on arrival at the variable medium. It will be sufficient to consider the case where m is real, that is, where the change of density is but moderately rapid.

By supposition, there is no negative wave in the variable medium, so that $A=0$ in (2). Thus

$$y=Bx^{\frac{1}{2}-im}, \quad \frac{dy}{dx}=(\tfrac{1}{2}-im)\,Bx^{-\frac{1}{2}-im};$$

and, when $x=x_1$, $$\frac{dy}{y\,dx}=\frac{\frac{1}{2}-im}{x_1} \quad \dots\dots\dots\dots\dots\dots\dots\dots(5).$$

The general solution for the uniform medium may be written

$$y=H\,e^{-in\frac{x-x_1}{x_1}}+K\,e^{+in\frac{x-x_1}{x_1}} \quad \dots\dots\dots\dots\dots\dots(6),$$

from which, when $x=x_1$,

$$\frac{dy}{y\,dx}=-\frac{in}{x_1}\,\frac{H-K}{H+K} \quad \dots\dots\dots\dots\dots\dots\dots\dots(7).$$

In equation (6), H represents the amplitude of the incident positive wave, and K the amplitude of the reflected negative wave. The condition to be satisfied at $x=x_1$ is expressed by equating the values of $\frac{dy}{y\,dx}$ given by (5) and (7). Thus

$$\frac{K}{H}=\frac{i(n-m)+\frac{1}{2}}{i(n+m)-\frac{1}{2}} \dots\dots\dots\dots\dots\dots\dots\dots(8),$$

which gives, in symbolical form, the ratio of the reflected to the incident vibration.

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* Theory of Sound, § 141.

Having regard to (3), we may write (8) in the form

$$\frac{K}{H} = \frac{e^{i\gamma}}{2(n+m)} \quad\text{.................................}(9);$$

so that the real amplitude of the reflected wave is $\frac{1}{2}(n+m)^{-1}$ of that of the incident. Thus, as was to be expected, when n and m are great, *i.e.*, when the density changes slowly in the variable medium, there is but little reflection.

Passing on now to the more important problem, we will suppose that the variable medium extends only so far as the point $x=x_2$, beyond which the density retains uniformly its value at that point. A positive wave travelling at first in a uniform medium of density proportional to x_1^{-2}, passes at the point $x=x_1$ into a variable medium of density proportional to x^{-2}, and again, at the point $x=x_2$, into a uniform medium of density proportional to x_2^{-2}. The velocities of propagation are inversely proportional to the square roots of the densities, *i.e.*, vary as x, so that, if μ be the refractive index between the extreme media,

$$\mu = \frac{x_1}{x_2} \quad\text{...}(10).$$

The thickness (d) of the layer of transition is

$$d = x_2 - x_1 \quad\text{.......................................}(11).$$

The wave-lengths in the two media are given by

$$\lambda_1 = \frac{2\pi x_1}{n}, \quad \lambda_2 = \frac{2\pi x_2}{n};$$

so that

$$n = \frac{2\pi d}{\lambda_2 - \lambda_1} = \frac{2\pi d}{(\mu^{-1}-1)\lambda_1} \quad\text{...........................}(12).$$

For the first medium we take, as before,

$$y = He^{-in\frac{x-x_1}{x_1}} + Ke^{+in\frac{x-x_1}{x_1}} \quad\text{..................}(6),$$

giving, when $x = x_1$,

$$\frac{dy}{y\,dx} = -\frac{in}{x_1}\,\frac{H-K}{H+K} = -\frac{in\theta}{x_1} \quad\text{..................}(7),$$

if, for brevity, we write θ for $\frac{H-K}{H+K}$.

For the variable medium,

$$y = Ax^{\frac{1}{2}+im} + Bx^{\frac{1}{2}-im} \quad\text{........................}(2),$$

giving, when $x = x_1$,

$$\frac{dy}{y\,dx} = x_1^{-1}\,\frac{(\frac{1}{2}+im)\,Ax_1^{im} + (\frac{1}{2}-im)\,Bx_1^{-im}}{Ax_1^{im} + Bx_1^{-im}} \quad\text{...............}(13).$$

Hence the condition to be satisfied at $x = x_1$ gives

$$\tfrac{1}{2} + im \frac{Ax_1^{im} - Bx_1^{-im}}{Ax_1^{im} + Bx_1^{-im}} = -\,in\theta\,;$$

whence

$$\frac{A}{B} = x_1^{-2im} \frac{im - in\theta - \frac{1}{2}}{im + in\theta + \frac{1}{2}} \quad \text{..........................(14).}$$

The condition to be satisfied at $x = x_2$ may be deduced from (14), by substituting x_2 for x_1, putting at the same time $\theta = 1$ in virtue of the supposition that in the second medium there is no negative wave. Hence, equating the two values of $A : B$, we get

$$x_1^{-2im} \frac{im - in\theta - \frac{1}{2}}{im + in\theta + \frac{1}{2}} = x_2^{-2im} \frac{im - in - \frac{1}{2}}{im + in + \frac{1}{2}} \quad \text{..................(15),}$$

as the equation from which the reflected wave in the first medium is to be found. Having regard to (3), we get

$$\theta = \frac{H - K}{H + K} = \frac{m + n + \frac{1}{2}i + \mu^{2im}(m - n - \frac{1}{2}i)}{m + n - \frac{1}{2}i + \mu^{2im}(m - n + \frac{1}{2}i)},$$

so that

$$\frac{K}{H} = \frac{-i + \mu^{2im}\, i}{2(m+n) + 2\mu^{2im}(m - n)} \quad \text{......................(16).}$$

This is the symbolical solution. To interpret it in real quantities, we must distinguish the cases of m real and m imaginary. If the transition be not too sudden, m is real, and (16) may be written

$$\frac{K}{H} = \frac{i}{2} \, \frac{-1 + \cos(2m \log \mu) + i \sin(2m \log \mu)}{m + n + (m - n)\cos(2m \log \mu) + i(m - n)\sin(2m \log \mu)}.$$

Thus the expression for the ratio of the *intensities* of the reflected and incident waves is, after reduction,

$$\frac{\sin^2(m \log \mu)}{4m^2 + \sin^2(m \log \mu)} \quad \text{..............................(17).}$$

If m be imaginary, we may write $im = m'$; (16) then gives, for the ratio of intensities,

$$\frac{(\mu^{m'} - \mu^{-m'})^2}{(\mu^{m'} - \mu^{-m'})^2 + 16m'^2} \quad \text{..............................(18);}$$

or, if we introduce the notation of hyperbolic trigonometry,

$$\frac{\sinh^2(m' \log \mu)}{\sinh^2(m' \log \mu) + 4m'^2} \quad \text{..............................(19).}$$

For the critical value $m = 0$, we get, from (17) or (19),

$$\frac{(\log \mu)^2}{4 + (\log \mu)^2} \quad \text{....................................(20).}$$

These expressions allow us to trace of the effect of a more or less gradual transition between media of given index. If the transition be absolutely abrupt, $n = 0$, by (12); so that $m' = \frac{1}{2}$. In this case,

(18) gives us Young's well-known formula

$$\left(\frac{\mu-1}{\mu+1}\right)^2 \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(21).$$

Since $\frac{\sinh x}{x}$ increases continually from $x=0$, the ratio (19) increases continually from $m'=0$ to $m'=\frac{1}{2}$, *i.e.*, diminishes continually from the case of sudden transition $m'=\frac{1}{2}$, when its value is (21), to the critical case $m'=0$, when its value is (20), after which this form no longer holds good. When $m'=0$, $n=\frac{1}{2}$, and, by (12), $d=\frac{\lambda_1-\lambda_2}{4\pi}$.

When $n>\frac{1}{2}$, (17) is the appropriate form. We see from it that, as n increases, the reflection diminishes, until it vanishes, when $m\log\mu=\pi$, *i.e.*, when

$$n^2=\tfrac{1}{4}+\frac{\pi^2}{(\log\mu)^2} \quad\dots\dots\dots\dots\dots\dots\dots\dots(22).$$

With a still more gradual transition the reflection revives, reaches a maximum, again vanishes when $m\log\mu=2\pi$, and so on.

As a numerical example I will take the case of $\mu=\frac{2}{3}$, applicable to the reflection of luminous waves in passing from glass to air. The value of $\log_e\frac{3}{2}$ is ·40546520. For the sudden transition, we have, from (21), ·040. When $m'=0$, *i.e.*, when $d=\frac{\lambda_2-\lambda_1}{4\pi}$, the reflection is, by (20), ·039478, but very little less. For moderate values of m', we may derive, from (18),

$$\frac{(\log\mu)^2}{(\log\mu)^2+4}\left\{1+\frac{(\log\mu)^2}{(\log\mu)^2+4}\,\frac{4m'^2}{3}+\dots\right\} \quad\dots\dots\dots\dots(23);$$

or, in the present example,

$$\cdot039478\left(1+\frac{4m'^2}{3}\times\cdot039478\right).$$

There is thus no appreciable variation of reflection from $m'=\frac{1}{2}$ to $m'=0$. The reflection vanishes when

$$n^2=\tfrac{1}{4}+\frac{a^2\pi^2}{(\cdot4055)^2},$$

a being any integer. Even when $a=1$, the first term in the value of n^2 is small relatively to the second, and we may take, with sufficient approximation, $n=\frac{a\pi}{\cdot4055}$; or, by (12), $d=\frac{a\,(\lambda_2-\lambda_1)}{\cdot811}$.

The thickness of the layer of transition at which the reflection first vanishes is thus a little greater than the difference of the wave-lengths. The revivals of reflection which occur when the transition is still more gradual are not important. The approximate values are $\left(\frac{2}{3\pi}\right)^2$, $\left(\frac{2}{5\pi}\right)^2$, &c., the reflection with abrupt transition being taken as standard. Thus, at the first revival, the reflection is only about $\frac{1}{22}$ of its original value.

On the Stability, or Instability, of certain Fluid Motions. By Lord RAYLEIGH, F.R.S., Professor of Experimental Physics in the University of Cambridge.

[*Read February* 12*th*, 1880.]

In a former communication to the Society on the "Instability of Jets,"* I applied a method due to Sir W. Thomson, to calculate the manner of falling away from equilibrium of jets bounded by one or more surfaces of discontinuity. Such interest as these investigations possessed was due principally to the possibility of applying their results to the explanation of certain acoustical phenomena relating to sensitive flames and smoke jets. But it soon appeared that in one important respect the calculations failed to correspond with the facts.

To fix the ideas, let us take the case of an originally plane surface of separation, on the two sides of which the fluid moves with equal and opposite constant velocities ($\pm V$). In equilibrium, the elevation h, at every point x along the surface, is zero. It is proved that, if initially the surface be a rest in the form defined by $h = H \cos \kappa x$, then, after a time t, its form is given by

$$h = H \cos \kappa x \cosh \kappa V t \text{.........................}(1),$$

provided that, throughout the whole time contemplated, the disturbance is small. In the same sense as that in which the frequency of vibration measures the stability of a system vibrating about a configuration of stable equilibrium, so the coefficient κV of t, in equations such as (1), measures the instability of an unstable system; and we see, in the present case, that the instability increases without limit with κ; that is to say, the shorter the wave-length of the sinuosities on the surface of separation, the more rapidly are they magnified.

The application of this result to sensitive jets would lead us to the conclusion that their sensitiveness increases indefinitely with pitch. It is true that, in the case of certain flames, the pitch of the most efficient sounds is very high, not far from the upper limit of human hearing; but there are other kinds of sensitive jets on which these high sounds are without effect, and which require for their excitation a moderate or even a grave pitch.

A probable explanation of the discrepancy readily suggests itself. The calculations are founded upon the supposition that the changes of velocity are discontinuous—a supposition which cannot possibly agree with reality. In consequence of fluid friction, a surface of discontinuity, even if it could ever be formed, would instantaneously disappear, the transition from the one velocity to the other becoming more and more gradual, until the layer of transition attained a sensible width. When this width is comparable with the wave-length of the sinuosity, the solution for an abrupt transition ceases to be applicable, and we have no reason for supposing that the instability would increase for much shorter wave-lengths.

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* "Proceedings," Vol. x., p. 4, Nov. 14, 1878.

In the following investigations, I shall suppose that the motion is entirely in two dimensions, parallel (say) to the plane xy, so that (in the usual notation) w is zero, as well as the rotations ξ, η. The rotation ζ parallel to z is connected with the velocities u, v by the equation

$$\zeta = \tfrac{1}{2}\left(\frac{du}{dy} - \frac{dv}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When the phenomena under consideration are such that the compressibility may be neglected, the condition that no fluid is anywhere introduced or abstracted, gives

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

In the absence of friction, ζ remains constant for every particle of the fluid; otherwise, if ν be the kinematic viscosity, the general equation for ζ is

$$\frac{\delta\zeta}{\delta t} = \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz} + \nu\nabla^2\zeta \ldots\ldots\ldots\ldots\ldots(4),*$$

where

$$\frac{\delta}{\delta t} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

For the proposed applications to motion in two dimensions, these equations reduce to

$$\frac{\delta\zeta}{\delta t} = \nu\nabla^2\zeta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$\frac{\delta}{\delta t} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

while the two other equations similar to (4) are satisfied identically.

In order to investigate the influence of friction on stratified motion, we may now suppose that v is zero, while u and ζ are functions of y only. Our equations then give simply

$$\frac{d\zeta}{dt} = \nu\frac{d^2\zeta}{dy^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

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* Lamb's Motion of Fluids, p. 243.

which shows that the rotation ζ is conducted according to precisely the same laws as heat. In the case of air at atmospheric pressure, the value of ν is, according to Maxwell's experiments,

$$\nu = \cdot 16,^*$$

not differing greatly from the number (·22) corresponding to the conduction of temperature in *iron.*

The various solutions of (10), discovered by Fourier, are at once applicable to our present purpose. In the problem already referred to, of a surface of discontinuity $y = 0$, separating portions of fluid moving with different but originally constant velocities, the rotation is at first zero, except upon the surface itself, but it is rapidly diffused into the adjacent fluid. At time t its value at any point y is

$$\zeta = \int_{-\infty}^{+\infty} \zeta\, dy \,.\, \frac{e^{-\frac{y^2}{4\nu t}}}{2\sqrt{(\pi\nu t)}} \dots\dots\dots\dots(11),$$

and $$\int_{-\infty}^{+\infty} \zeta\, dy = \tfrac{1}{2}\int \frac{du}{dy}\, dy = \tfrac{1}{2}\,(V_2 - V_1) \quad \dots\dots\dots\dots(12),$$

if V_2, V_1 are the velocities on the positive and negative sides of the surface respectively. If $y^2 = 4\nu t$, the value of ζ is less than that to be found at $y = 0$, in the ratio $e : 1$. Thus, after a time t, the thickness of the layer of transition $(2y)$ is comparable in magnitude with $1{\cdot}6\sqrt{t}$; for example, after one second it may be considered to be about $1\frac{1}{2}$ centimetres. In the case of water, the coefficient of conductivity is much less. It seems that $\nu = \cdot 014$; so that, after one second, the layer is about half a centimetre thick.

The circumstances of a two-dimensional jet will be represented by supposing the velocity to be limited initially to an infinitely thin layer at $y = 0$. It is convenient here to use the velocity u itself instead of ζ.

Since $\zeta = \frac{1}{2}\frac{du}{dy}$, $$\frac{du}{dt} = \nu\,\frac{d^2u}{dy^2} \dots\dots\dots\dots(13),$$

and thus the solution is of the same form as before:

$$u = \int_{-\infty}^{+\infty} u\, dy \,.\, \frac{e^{-\frac{y^2}{4\nu t}}}{2\sqrt{(\pi\nu t)}} \dots\dots\dots\dots(14).$$

We may conclude that, however thin a jet of air may be initially, its thickness after one second is comparable with $1\frac{1}{2}$ centimetres. A similar calculation may be made for the case of a linear jet, whose whole velocity is originally concentrated in one line.

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* The centimetre and second being units.

There is, therefore, ample foundation for the opinion that the phenomena of sensitive jets may be greatly influenced by fluid friction, and deviate materially from the results of calculations based upon the supposition of discontinuous changes of velocity. Under these circumstances, it becomes important to investigate the character of the equilibrium of stratified motion in cases more nearly approaching what is met with in practice. Fully to include the effects of friction, would immensely increase the difficulties of the problem. For the present, at least, we must treat the fluid as frictionless, and be satisfied if we can obtain solutions for laws of stratification, which are free from discontinuity. For the undisturbed motion, the component velocity v is zero, and u is a function of y only. A curve in which u is ordinate and y is abscissa, represents the law of stratification, and may be called, for brevity, the velocity curve.

A class of problems which can be dealt with by fairly simple methods, is obtained by supposing the rotation ζ to be constant throughout layers of finite thickness, and only to change its value in passing a limited number of planes for which y is constant. In such cases, the velocity curve is composed of portions of straight lines which meet one another at finite angles. This state of things may be supposed to be slightly disturbed by bending the surfaces of transition, and the determination of the subsequent motion depends upon that of the form of these surfaces. For ζ retains its constant value throughout each layer unchanged in the absence of friction, and, by a well-known theorem, the whole motion depends upon ζ. We shall suppose that the functions deviate from their equilibrium values by quantities proportional to $e^{i\kappa x}$, so that everything is periodic with respect to x in a distance λ equal to $2\pi\kappa^{-1}$. By Fourier's theorem, the solution may be generalised sufficiently to cover the case of an arbitrary deformation of the surfaces. As functions of the time, the disturbances will be assumed to be proportional to e^{int}, where n may be either real or complex, and the character of the resulting motion is determined in great measure by the value of n, found by the solution of the problem.

By a theorem due to Helmholtz, the effect of any element dA rotating with angular velocity ζ, is to produce, at a point whose distance from the element is r, a transverse velocity q, such that

$$q = \frac{\zeta dA}{\pi r} \quad \text{.......................................(15).}$$

In the application of this result to the problems in hand, it will be convenient to regard the actual value of ζ at time t as made up of two parts, (1) its undisturbed value, (2) the difference between its actual value and (1). The effect of (1) is to produce the undisturbed system of velocities, on which the small effect of (2) is superposed; and the calculation of the latter effects evidently involves integrations which extend only over the infinitely small areas included between the disturbed and undisturbed surfaces of transition. Suppose that the equation of one of these surfaces, reckoned from its undisturbed position, is

$$\eta = He^{i\kappa\xi} \qquad\qquad (16),$$

in which H is not necessarily real. Then $dA = \eta\, d\xi$, and if $\Delta\zeta$ be the excess of the value of ζ on the upper above that on the lower side of the surface, we get, by (15), at any point whose abscissa is x and distance from the surface is b,

$$q = -\frac{\Delta\zeta\, .\, \eta\, d\xi}{\pi r} \qquad\qquad (17),$$

where

$$r^2 = b^2 + (\xi - x)^2 \qquad\qquad (18).$$

The velocity q, given by (17), is perpendicular to r. The next step, previous to integration, is to resolve it in the fixed directions of x and y. The resolution is effected by introduction of the factors b/r, and $(\xi - x)/r$; and thus, for the whole effect of the surface under consideration, $u = -\frac{b\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{\eta\, d\xi}{r^2}$, $v = -\frac{\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{\eta\,(\xi - x)\, d\xi}{r^2}$;

or, by (16),

$$u = -\frac{bH\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\kappa\xi}\, d\xi}{b^2 + (\xi - x)^2} \qquad\qquad (19),$$

$$v = -\frac{H\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\kappa\xi}\,(\xi - x)\, d\xi}{b^2 + (\xi - x)^2} \qquad\qquad (20).$$

The integrals are readily evaluated by the theorem

$$\int_{-\infty}^{+\infty}\frac{\cos a x}{1 + x^2}\, dx = \int_{-\infty}^{+\infty}\frac{x \sin a x}{1 + x^2}\, dx = \pi e^{-a};$$

and we obtain

$$\left.\begin{aligned} u &= -\ H\Delta\zeta\, e^{i\kappa x} e^{-\kappa b} \\ v &= -\, iH\Delta\zeta\, e^{i\kappa x} e^{-\kappa b} \end{aligned}\right\} \qquad\qquad (21).$$

In the derivation of (21), η has been treated as infinitesimal in comparison with b, but in the sequel we shall require to apply the formula to points situated upon the surface itself. The value of u would need more careful examination, but that of v is easily seen to be equally applicable when b is zero, since the neighbouring elements do not contribute sensibly to the value of the integral. In fact, the value is the same on whichever side of the surface the point under consideration is situated, and b is in both cases to be taken positive. Accordingly, when b is zero, we are to take simply

$$v = -\, iH\Delta\zeta e^{i\kappa x} \qquad\qquad (22).$$

We are now prepared to enter upon the consideration of special problems. As a first example, let us suppose that on the upper side of a layer of thickness b the undisturbed velocity u is equal to $+V$, and on the lower side to $-V$, while inside the layer it changes uniformly. Thus

$$\zeta = \tfrac{1}{2}\frac{du}{dy} = \frac{V}{b} \quad \text{.......................................(23)}$$

inside the layer, and outside the layer $\zeta = 0$. In the disturbed motion, let the equations of the upper and lower surfaces be respectively, at time t,

$$\left.\begin{aligned} \eta &= H e^{int} e^{i\kappa x} \\ \eta' &= H' e^{int} e^{i\kappa x} \end{aligned}\right\} \quad \text{.......................................(24)} ;$$

then, by (21), (22), (23), the whole value of v for a point on the upper surface is

$$v = ib^{-1} V e^{int} e^{i\kappa x} (H - H' e^{-\kappa b}) \quad \text{..........................(24)}',$$

and for the lower surface

$$v = ib^{-1} V e^{int} e^{i\kappa x} (H e^{-\kappa b} - H') \quad \text{..........................(25)}.$$

From these values of v the position of the surfaces at time $t+dt$ may be calculated. At time t, η corresponds to x; at time $t+dt$, $\eta + vdt$ corresponds to $x + udt$, u being the whole component velocity parallel to x. Thus, at time $t+dt$, corresponding to x, we have

$$\eta + vdt - \frac{d}{dx}(\eta + vdt) \, . \, udt ;$$

or, on neglecting the squares of small quantities,

$$\eta + \left(v - V\frac{d\eta}{dx}\right) dt.$$

Now, from (24),

$$\frac{d\eta}{dt} = in\eta ;$$

so that, equating the two values of $\frac{d\eta}{dt}$, we get, from (24),

$$inH = ib^{-1} V (H - H' e^{-\kappa b}) - i\kappa V H,$$

or

$$\left(\frac{nb}{V} - 1 + \kappa b\right) H + e^{-\kappa b} H' = 0 \quad \text{..........................(26)}.$$

In like manner, by considering the motion of the lower surface, we get

$$e^{-\kappa b} H + \left(-\frac{nb}{V} - 1 + \kappa b\right) H' = 0 \quad \text{.....................(27)}.$$

By eliminating the ratio $H' : H$ between (26) and (27), we obtain, as the equation giving the admissible values of n,

$$n^2 = \frac{V^2}{b^2}\left\{(\kappa b - 1)^2 - e^{-2\kappa b}\right\} \quad \text{..........................(28)}.$$

When κb is small, that is, when the wave-length is great in comparison with b, the case approximates to that of a sudden transition. Thus

$$n^2 = \frac{V^2}{b^2}\{1-2\kappa b+\kappa^2b^2-(1-2\kappa b+2\kappa^2b^2+\ldots)\}$$

$$= -\kappa^2V^2 \text{ approximately} \qquad\qquad (29),$$

in agreement with equation (30) of my former paper. In this case the motion is unstable. On the other hand, when κb is great, we find, from (28),

$$n^2 = \kappa^2V^2 \qquad\qquad (30);$$

and, since the values of n are real, the motion is *stable.* It appears, therefore, that so far from the instability increasing indefinitely with diminishing wave-length, as when the transition is sudden, a diminution of wave-length below a certain value entails an instability which gradually decreases, and is finally exchanged for actual stability. The following table exhibits more in detail the progress of $b^2V^{-2}n^2$ as a function of κb:—

κb	$b^2V^{-2}n^2$	κb	$b^2V^{-2}n^2$
·2	−·03032	1·0	−·13534
·4	−·08933	1·2	−·05072
·6	−·14120	1·3	+·01573
·8	−·16190	2·0	+·98168

We see that the instability is greatest when $\kappa b = {}$·8 nearly, that is, when $\lambda = 8b$; and that the passage from instability to stability takes place when $\kappa b = 1·3$ nearly, or when $\lambda = 5b$.

Corresponding with the two values of n, there are two ratios of $H' : H$ determined by (26) or (27), each of which gives a normal mode of disturbance, and by means of these normal modes the results of an arbitrary displacement of the two surfaces may be represented. It will be seen that for the stable disturbances the ratio $H' : H$ is real, indicating that the sinuosities of the two surfaces are at every moment in the same phase.

We may next take an example of a jet of thickness $2b$ moving in still fluid, supposing that the velocity in the middle of the jet is V, and that it falls uniformly to zero on either side. Taking the middle line as axis of x, we may write

$$u = V\left(1 \mp \frac{y}{b}\right) \qquad\qquad (31),$$

in which the − sign applies to the upper, and the + sign to the lower half of the jet. Thus

$$\zeta = \tfrac{1}{2}\frac{du}{dy} = \mp\tfrac{1}{2}\frac{V}{b} \qquad\qquad (32)$$

within the jet, and outside the jet $\zeta = 0$. In this problem there are three surfaces to be considered. We will suppose the equation of the upper surface, for which $\Delta\zeta = \frac{V}{2b}$, to be $\eta = He^{i\kappa x}e^{int}$; that of the middle surface, for which $\Delta\zeta = -\frac{V}{b}$, to be $\eta' = H'e^{i\kappa x}e^{int}$; and that of the lower surface, for which $\Delta\zeta = \frac{V}{2b}$, to be $\eta'' = H''e^{i\kappa x}e^{int}$.

From (22), the velocities v are to be calculated as before. We find

$$v\ (\text{upper surface}) = \frac{iV}{2b}e^{int}e^{i\kappa x}\{-H+2e^{-\kappa b}H'-e^{-2\kappa b}H''\},$$

$$v\ (\text{middle surface}) = \frac{iV}{2b}e^{int}e^{i\kappa x}\{-e^{-\kappa b}H+2H'-e^{-\kappa b}H''\},$$

$$v\ (\text{lower surface}) = \frac{iV}{2b}e^{int}e^{i\kappa x}\{-e^{-2\kappa b}H+2e^{-\kappa b}H'-H''\}.$$

For the upper and lower surfaces the horizontal velocity is zero, and for the middle surface it is V. In the same manner as for (26), we thus obtain

$$\left.\begin{aligned} mH-2\gamma H'+\gamma^2H''=0 \\ \gamma H+(m+2\kappa b-3)\,H'+\gamma H''=0 \\ \gamma^2H-2\gamma H'+mH''=0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

in which γ is written for $e^{-\kappa b}$, and m is written $2bV^{-1}n+1$. The elimination from (33) of $H:H':H''$ gives the following cubic in m:—

$$m^3+(2\kappa b-3)\,m^2+\gamma^2(4-\gamma^2)\,m-\gamma^4(1+2\kappa b)=0\ldots\ldots\ldots\ldots(34).$$

By inspection of (33), we see that one of the normal disturbances is defined by $H'=0$, $H+H''=0$, and that the corresponding value of m is γ^2. It follows that $m-\gamma^2$ is a factor of the cubic expression in (34), and the remaining quadratic factor is readily obtained by division. Thus (34) assumes the form

$$(m-\gamma^2)\{m^2+(2\kappa b-3+\gamma^2)\,m+\gamma^2(1+2\kappa b)\}=0\ \ldots\ldots\ldots(35).$$

For the symmetrical disturbance

$$n=-\frac{V}{2b}(1-e^{-2\kappa b})\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

a *real* quantity, indicating that the motion is *stable* so far as this mode of disturbance is concerned.

The other two values of n are real, if

$$(2\kappa b-3+\gamma^2)^2-4\gamma^2(1+2\kappa b)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

be positive, but not otherwise. When κb is infinite, $\gamma=0$, and (37) reduces to $4\kappa^2b^2$, which is positive; so that the motion is stable when the wave-length is small in comparison with the thickness of the jet. On the other hand, as may readily be proved by expanding γ in (37), the motion is unstable, when the wave-length is great in comparison with the thickness of the jet. The values of (37) can be more easily computed when it is thrown into the form

$$(5+2\kappa b-e^{-2\kappa b})^2-16\,(1+2\kappa b)\ \ldots\ldots\ldots\ldots\ldots\ldots(38).$$

Some corresponding values of (38) and $2\kappa b$ are shown below :—

$2\kappa b$	(38)	$2\kappa b$	(38)
·5	−·054	2·5	−·975
1·0	−·279	3·0	−·794
1·5	−·599	3·5	−·263
2·0	−·876	4·0	+·671

The wave-length of maximum instability is about $2\frac{1}{2}$ times the thickness ($2b$) of the jet; while, for a wave-length about half as great again, or more, the motion becomes stable.

Although it is the fact, as I have found by experiment, that a sensitive jet breaks up by becoming sinuous as a whole, the result that a symmetrical mode of disturbance is stable, is special to the law of velocity assumed in the foregoing example. In order to illustrate this, I will state the results for the more general law of velocity obtained by supposing the maximum velocity V to extend through a layer of finite thickness b' in the middle of the jet. The rotation ζ is zero in this central layer; in the adjacent layers of thickness b, $\zeta=\mp\frac{V}{2b}$, as before. The equations of the four surfaces, in crossing which ζ changes its value, being represented by

$$\eta,\ \eta',\ \eta'',\ \eta'''=(H,\ H',\ H'',\ H''')\,e^{int}e^{i\kappa x},$$

we may obtain four equations involving n and the three ratios $H:H':H'':H'''$. The elimination of these ratios gives a biquadratic in n, which, however, is easily split into two quadratics, one of which relates to symmetrical disturbances, for which $H+H'''=0$, $H'+H''=0$; and the other to disturbances for which $H-H'''=0$, $H'-H''=0$. The resulting equation in n is

$$\left(\frac{2bn}{V}\right)^2+(\pm\gamma'\mp\gamma'\gamma^2+2\kappa b)\,\frac{2bn}{V}$$

$$\pm\gamma'-1+2\kappa b+\gamma^2\,(1\mp\gamma'\mp 2\kappa b\gamma')=0\ \ldots\ldots(39),$$

γ' being written for $e^{-\kappa b'}$. In (39) the upper signs correspond to the symmetrical displacements. The roots are real, and the disturbances are stable, if

$$(\pm\gamma'\mp\gamma'\gamma^2+2\kappa b)^2-4\,[\pm\gamma'-1+2\kappa b+\gamma^2\,(1\mp\gamma'\mp 2\kappa b\gamma')]\ \ldots(40)$$

be positive.

In what follows, we will limit our attention to the symmetrical disturbances, that is, to the upper signs in (40), and to terms of orders not higher than the first in b'. The expression (40) may then be reduced to $(1-\gamma^2-2\kappa b)^2+2\kappa b'(1+\gamma^2)(1-\gamma^2-2\kappa b)$ ....(41).

If κb be very small, this becomes

$$4\kappa^4b^4-8\kappa b'.\kappa^2b^2 \quad\text{..............................}(42).$$

If b' be zero, (42) is positive, and the disturbance is stable, as we found before; but if b and b' be of the same order of magnitude, and both very small compared to λ, it follows from (42) that the disturbance is unstable.

If, in (39), we suppose that b is zero, we fall back upon the case of a jet of uniform velocity V and thickness b' moving in still fluid. The equation for n, after division by b^2, becomes

$$n^2+(1\pm\gamma')\kappa V.n+\tfrac{1}{2}(1\pm\gamma')\kappa^2V^2=0,$$

or

$$(n+\kappa V)^2\frac{1\pm\gamma'}{1\mp\gamma'}+n^2=0 \quad\text{..........................}(43).$$

In the notation of my former paper, $b'=\tfrac{1}{2}l$, so that

$$\frac{1+\gamma'}{1-\gamma'}=\coth\kappa l,\quad \frac{1-\gamma'}{1+\gamma'}=\tanh\kappa l;$$

and the equations there numbered (48) and (55) agree with (43).

Another particular case of (39), comparable with previous results, is obtained by supposing b' to be infinite.

I now pass to the consideration of certain cases in which the moving layers are bounded by fixed walls, instead of by an unlimited expanse of stationary fluid. The effect of the walls may be imitated by the introduction of an unlimited number of similar layers, in the same way as the vibrations of a string fixed at two points are often deduced from the theory applicable to an unlimited string. The displacements of the surfaces at which ζ changes its value being taken equal and opposite in consecutive layers, the value of v, at the places occupied by the walls, is, by symmetry, zero; and thus the presence or absence of the actual walls is a matter of indifference.

Let us first suppose that the distribution of velocity within the layer is that given in (31), uniformly increasing from zero at the walls to a maximum V in the middle, the distance between the walls being $2b$. The actual surface of transition and its successive images make contributions to the value of v at the surface, which are alternately opposite in sign, and, as regards numerical value, form a geometrical progression with common ratio $e^{-2\kappa b}$ or γ^2. Thus, with the same notation as before, we have at the surface, from (21), (22),

$$v = \frac{iVH}{b} e^{int} e^{i\kappa x} \{1 - 2(\gamma^2 - \gamma^4 + \gamma^6 - \dots)\}$$

$$= \frac{iVH}{b} e^{int} e^{i\kappa x} \cdot \frac{1-\gamma^2}{1+\gamma^2};$$

so that, as in previous problems,

$$n = -\kappa V + \frac{V}{b} \frac{1-\gamma^2}{1+\gamma^2};$$

or, as it may also be written,

$$\frac{bn}{V} = \tanh \kappa b - \kappa b \dots\dots (44).$$

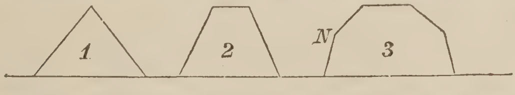
If there be a layer of finite width b' in the centre, throughout which the undisturbed velocity is V, we obtain

$$\left.\begin{aligned} \left(2\frac{nb}{V} + 2\kappa b - \frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'}\right) H - \frac{\gamma' - \gamma^2}{1+\gamma^2\gamma'} H' = 0 \\ -\frac{\gamma' - \gamma^2}{1+\gamma^2\gamma'} H + \left(2\frac{nb}{V} + 2\kappa b - \frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'}\right) H' = 0 \end{aligned}\right\} \dots\dots (45),$$

in which H, H' refer to the two surfaces of transition, and $\gamma' = e^{-\kappa b'}$. Equation (45) shews, as might also be inferred from symmetry, that $H \pm H' = 0$, while

$$\frac{2nb}{V} + 2\kappa b - \frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'} = \pm \frac{\gamma' - \gamma^2}{1+\gamma^2\gamma'} \dots\dots (46).$$

Since the values of n in (46) are real, the disturbance is *stable*.



In these examples the velocity curves are those represented by figs. (1) and (2). I have taken a further step in the direction of generalisation by calculating the motion for a velocity curve in the form of (3). The criterion of stability is complicated in its expression, but it is not difficult to shew that the motion is stable if the angle N be a projecting angle. From these examples there seemed to be some reason for thinking that the motion would be stable, whenever the velocity curve was of one curvature throughout; and this led me to attack the question by a more general method, which I will now explain.

Let us suppose that the conditions of steady motion are satisfied by $u = U$, $v = V$, $\zeta = Z$; and let us trace the effects of superposing upon this motion a disturbance for which $u = \delta u$, $v = \delta v$, $\zeta = \delta\zeta$. Both the original motion and the disturbance satisfy the equation of continuity (3).

Since, in the absence of friction, the rotation of every element remains unchanged, $\frac{d(Z+\delta\zeta)}{dt}=0,$

or $$\frac{d}{dt}(Z+\delta\zeta)+(U+\delta u)\frac{d}{dx}(Z+\delta\zeta)+(V+\delta v)\frac{d}{dy}(Z+\delta\zeta)=0.$$

This equation is satisfied by supposition, if δu, δv, $\delta\zeta=0$. If we omit the squares and products of the small quantities, it becomes

$$\frac{d\delta\zeta}{dt}+U\frac{d\delta\zeta}{dx}+V\frac{d\delta\zeta}{dy}+\delta u\frac{dZ}{dx}+\delta v\frac{dZ}{dy}=0 \quad \text{............(47).}$$

If $V=0$, and U be a function of y only, (47) reduces to

$$\frac{d\delta\zeta}{dt}+U\frac{d\delta\zeta}{dx}+dv\frac{dZ}{dy}=0\,;$$

or, since in this case $Z=\frac{1}{2}\frac{dU}{dy}$,

$$\left(\frac{d}{dt}+U\frac{d}{dx}\right)\left(\frac{d\delta u}{dy}-\frac{d\delta v}{dx}\right)+\frac{d^2U}{dy^2}\delta v=0 \quad \text{............(48).}$$

We now introduce the supposition that, as functions of x, δu and δv are proportional to $e^{i\kappa x}$, so that, by (3),

$$i\kappa\delta u+\frac{d\delta v}{dy}=0 \quad \text{..................................(49).}$$

We thus obtain, by elimination of δu,

$$\left(\frac{1}{i\kappa}\frac{d}{dt}+U\right)\left(\frac{d^2\delta v}{dy^2}-\kappa^2\delta v\right)-\frac{d^2U}{dy^2}\delta v=0 \quad \text{..............(50).}$$

If we further suppose that, as a function of t, δv is proportional to e^{int}, where n is a real or complex constant, we get

$$\left(\frac{n}{\kappa}+U\right)\left(\frac{d^2\delta v}{dy^2}-\kappa^2\delta v\right)-\frac{d^2U}{dy^2}\delta v=0 \quad \text{.................(51).}$$

On this equation the solution of the special problems already considered may be founded. If, throughout any layer, the rotation Z be constant, $\frac{d^2U}{dy^2}=0$, and, wherever $n+\kappa U$ is not equal to zero, (51) reduces to

$$\frac{d^2\delta v}{dy^2}-\kappa^2\delta v=0 \quad \text{..............................(52).}$$

Equation (52) may, in fact, be easily established independently, on the assumption that the rotation throughout the layer is the same after disturbance as before. From (2),

$$\frac{d\zeta}{dx}=\frac{1}{2}\left(\frac{d^2u}{dx\,dy}-\frac{d^2v}{dx^2}\right)=-\frac{1}{2}\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right),$$

by (3); so that, when ζ is constant, $\nabla^2 v = 0$. In like manner $\nabla^2 u = 0$. If $\delta v \propto e^{i\kappa x}$, (52) follows immediately.

The solution of (52) is

$$\delta v = Ae^{\kappa y} + Be^{-\kappa y} \quad\text{.................................(53)},$$

where A and B are constants, not restricted to be real. For each layer of constant ζ, a fresh solution with fresh arbitraries is to be taken, and the partial solutions are to be fitted together by means of the proper boundary conditions. The first of these conditions is evidently

$$\Delta\delta v = 0 \quad\text{.....................................(54)}.$$

The second may be obtained by integrating (51) across the boundary. Thus

$$\left(\frac{n}{\kappa} + U\right) . \Delta\left(\frac{d\delta v}{dy}\right) - \Delta\left(\frac{dU}{dy}\right) . \delta v = 0 \quad\text{..................(55)}.$$

At a fixed wall $\delta v = 0$.

The reader may apply this method to the problem whose solution is expressed in (44).

In cases where $\frac{d^2U}{dy^2} = 0$, the substitution of (52) for (51), or the corresponding supposition that ζ is unchanged by the disturbance, amounts to a limitation on the generality of the solution. Suppose, for example, that the motion takes place between two fixed walls, at each of which $\delta v = 0$. Under these circumstances (53) shews that $\delta v = 0$ throughout, or no disturbance is possible; and this is obviously true if no new rotation is introduced by the disturbance. In order to obtain a general solution, we must retain the factor $n + \kappa U$ in (51). For any value of y which gives $n + \kappa U = 0$, (52) need not be satisfied; and thus any value of $-\kappa U$ is an admissible value of n, satisfying all the conditions of the problem.

I will now inquire, under what conditions (51) admits of a solution with a complex value of n; or, in other words, under what conditions the steady motion is unstable, assuming that, for two finite or infinite values of y, $\delta v = 0$. Let $n \div \kappa = p + iq$, $\delta v = \alpha + i\beta$, where p, q, α, β are real. Substituting in (51), we get

$$\frac{d^2\alpha}{dy^2} + i\frac{d^2\beta}{dy^2} = \left[\kappa^2 + \frac{d^2U}{dy^2}\,\frac{p+U-iq}{(p+U)^2+q^2}\right](\alpha + i\beta);$$

or, on equating to zero the real and imaginary parts,

$$\frac{d^2\alpha}{dy^2} = \kappa^2\alpha + \frac{d^2U}{dy^2}\,\frac{(p+U)\,\alpha + q\beta}{(p+U)^2+q^2} \quad\text{..........................(56)},$$

$$\frac{d^2\beta}{dy^2} = \kappa^2\beta + \frac{d^2U}{dy^2}\,\frac{-q\alpha + (p+U)\,\beta}{(p+U)^2+q^2} \quad\text{........................(57)}.$$

Multiplying (56) by β, (57) by α, and subtracting, we get

$$\beta\frac{d^2\alpha}{dy^2}-\alpha\frac{d^2\beta}{dy^2}=\frac{d^2U}{dy^2}\frac{q(\alpha^2+\beta^2)}{(p+U)^2+q^2}=\frac{d}{dy}\left(\beta\frac{d\alpha}{dy}-\alpha\frac{d\beta}{dy}\right)\ldots\ldots(58).$$

At the limits δv, and therefore both α and β, are, by hypothesis, zero. Hence, integrating (58) between the limits, we see that q must be zero, if $\frac{d^2U}{dy^2}$ be of one sign throughout the range of integration; so that, if the velocity curve is either wholly convex or wholly concave for the space between two limits at which $\delta v=0$, the motion is thoroughly stable.* This result covers all the special problems of motion between walls previously investigated. Its application to jets, for which $\frac{d^2U}{dy^2}$ changes sign, leaves the question of stability or instability still open.

Another general result worth notice may be obtained from (51). Writing it in the form

$$\frac{d^2\delta v}{dy^2}=\left\{\kappa^2+\frac{\frac{d^2U}{dy^2}}{\frac{n}{\kappa}+U}\right\}\delta v,$$

we see that, if n is real, δv cannot pass from one zero value to another zero value, unless $\frac{d^2U}{dy^2}$ and $n+\kappa U$ be somewhere of contrary signs. Thus, if we suppose that U is positive, and $\frac{d^2U}{dy^2}$ negative throughout, and that V is the greatest value of U, we find that $n+\kappa V$ must be positive. For an example see the equation immediately preceding (44).

If the stream lines of the steady motion be concentric circles instead of parallel straight lines, the character of the problem is not greatly changed. It may be proved that, if the fluid move between two rigid concentric circular walls, the motion is stable, provided that in the steady motion the rotation either continually increases or continually decreases in passing outwards from the axis.

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* More generally, the same conclusion follows if the ratio $\frac{d\delta v}{dy}:\delta v$ has real values at both limits.

March 11*th,* 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The following gentlemen were elected members:—

Messrs. C. S. Peirce, Lecturer on Logic, Johns Hopkins University; Emery McClintock, Milwaukee, Wisconsin; Prof. E. B. Seitz, M.A., Kirksville, Missouri; and Mr. E. Temperley, M.A., Queens' College, Cambridge. Mr. W. J. Curran Sharp was admitted into the Society. Dr. Elling Holst, of Christiania, was present at the meeting.

The following communications were made:—

"Notes on a General Method of solving Partial Differential Equations of the First Order with several Dependent Variables:" Mr. H. W. Lloyd Tanner, M.A.

"Note on the Integral Solution of $x^2-2Py^2=-z^2$ or $\pm 2z^2$ in certain cases:" Mr. S. Roberts, F.R.S.

Notes (1) "On a Geometrical Form of Landen's Theorem with regard to a Hyperbolic Arc;" (2) "On a class of Closed Ovals whose axes possess the same property as two Fagnanian arcs of an Ellipse:" Mr. J. Griffiths, M.A.

The following presents were received:—

"Om Poncelet's betydning for Geometrien et bidrag til de moderngeometriske ideers udviklingshistorie," af Elling Holst; Christiania, 1878: from the Author.

"Lettres de Laplace, Borda, Fuss et Jean-Albert Euler" (Extrait du "Bulletin des Sciences Mathématiques et Astronomiques," 2^e Série, Tome iii., 1879).

"De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan," par M. G. Darboux. (Ditto.)

"Addition au mémoire sur les fonctions discontinues," par M. G. Darboux ("Annales Scientifiques de l'Ecole Normale Supérieure," Paris, pp. 195—202).

"Applicazione dei Principii della Meccanica Analitica a Problemi"—Note i., ii., iii., iv., di Alessandro Dorna, Torino 1879: presented by the Meteorological Office.

"Ueber die Erhaltung des Geschlechtes bei Zwei ein-eindeutig auf einander bezogenen Plan-curven," von H. Schubert, in Hamburg ("Math. Ann.," xvi., pp. 180—182).

"Proceedings of the Royal Society," Vol. xxx., No. 201.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iii., Oct. 1879; Paris, 1879.

"Atti della R. Accademia dei Lincei—Transunti," Fasc. 3°, Febbraio 1880, Vol. iv.; Roma, 1880.

"Beiblätter zu den Annalen der Physik und Chemie," Band iv., Stück 3; Leipzig, 1880.

"Mémoire sur les solutions singulières des équations aux dérivées partielles du premier ordre," par M. G. Darboux; Paris, 1880: from the Author.

"Études Cinématiques," par M. E. J. Habich; Paris, 1879.

"Bulletin de la Société Mathématique de France," Tome viii., No. 1; Paris, 1880.

"Johns Hopkins University Circulars," No. 3 (Number devoted to Mathematics and Physics); Baltimore, Feb. 1880.

"Monatsbericht," Dec. 1879; Berlin, 1880.

"Crelle," Band 89, 1es Heft; Berlin, 1880.

"Proceedings of the Royal Society of Edinburgh," Session 1878—1879, Vol. x., 103, pp. 1—314.

"Three approximate solutions of Kepler's Problem," by H. A. Howe, A.M. (Cincinnati Society of Natural History, Proceedings, pp. 205—210).

"Educational Times," March, 1880.

"Ueber unendliche lineare Punktmannigfaltigkeiten," von Georg Cantor (" Math. Ann.," Bd. xv.)

"Zur Theorie der Zahlentheoretischen Functionen," von Georg Cantor.

Notes on a General Method of Solving Partial Differential Equations of the First Order with several Dependent Variables. By H. W. Lloyd Tanner, M.A.

[*Read March* 11*th*, 1880.]

1. The following notes relate to a method of solving equations with several dependent variables, which is a generalization of the process employed in the case where only one dependent variable occurs. The extension to the more general equations herein considered is easy, but it is perhaps worth stating since it leads to this curious conclusion; that the solution of such equations depends upon the solution of an auxiliary system similar to that which is required in the integration of equations with one dependent variable, but involving differential coefficients of the second or higher orders.

When there is one dependent variable, z, we determine p_i $\left(\text{or } \frac{dz}{dx_i}\right)$ by means of equations

$$F_1 = 0,\ \ F_2 = 0,\ \dots\ F_n = 0,$$

each F being a function of the p's and the variables z, x. The form of the functions, F, is derived from certain equations obtained from the identities
$$\frac{dp_i}{dx_j}-\frac{dp_j}{dx_i}=0.$$
It so happens that the equations for the determination of F, may be expressed in a form, the theory of which is well known; but this form does not appear to be capable of extension to more general cases. When all the F's are determined, the p's are given by algebraical processes, and z is determined from an integrable equation
$$dz-\Sigma p_i\,dx_i=0.$$
This theory has been discussed at some length, "Proceedings of London Mathematical Society," Vol. ix., pp. 77, &c., and the F-equations there given are of the same general type as those hereinafter obtained.

2. In the case in which we have m dependent variables
$$z_1,\ z_2\ \dots\ z_m,$$
and n independent variables $x_1,\ x_2\ \dots\ x_n$,

the most obvious analogue to the process just described, would be to determine
$$\frac{dz_i}{dx_j}$$
by means of mn equations $F=0$,

and to determine the forms of F by relations derived from the identities
$$\frac{d}{dx_k}\cdot\frac{dz_i}{dx_j}=\frac{d}{dx_j}\cdot\frac{dz_i}{dx_k}.$$
Each of these identities gives a relation between the F's; so that these mn functions are the solutions of $\frac{1}{2}mn\,(n-1)$ differential equations, which are of a very simple form. Represent, for convenience, $\frac{dz_i}{dx_j}$ by ij, and consider the Jacobian
$$\frac{d\,(F_1\ \dots\ F_{mn})}{d\,(11\ \dots\ ij\ \dots\ mn)},$$
in the numerator of which are all the F's, in the denominator all the differential coefficients ij. If from this we form another Jacobian by substituting x_k for ij, a third by substituting x_j for ik, and express that the two Jacobians thus derived are identically equal, we obtain one of the equations for the determination of the F's, viz., the one derived from the identity
$$\frac{d}{dx_k}\cdot\frac{dz_i}{dx_j}=\frac{d}{dx_j}\cdot\frac{dz_i}{dx_k}.$$
If we suppose that the F's may involve $z_1\ \dots\ z_n$, we must replace $\frac{dF}{dx_i}$, say, by the sum
$$\frac{dF}{dx_i}+\Sigma_k\,(ki)\,\frac{dF}{dz_k},$$

and the effect upon the Jacobians is similar, viz., each is replaced by a sum of Jacobians. Judging from analogy, we infer that this increase in the number of terms does not increase the difficulty of solving the system for F.

If the equations for F could be solved, we should be able at once to solve any proposed system of equations in $\frac{dz_i}{dx_j}$. Amongst the F's we should include the proposed system, and then determine the other F's. From all of these we may, by algebraical transformations, obtain $\frac{dz_i}{dx_j}$ as explicit functions of $x_1, x_2 \dots x_n$, $z_1 \dots z_m$, and arbitrary parameters. These values make the system

$$dz_i - \frac{dz_i}{dx_1}\,dx_1 - \frac{dz_i}{dx_2}\,dx_2 - \dots - \frac{dz_i}{dx_n}\,dx_n = 0,$$
$$i = 1, 2 \dots m,$$

an integrable system, from which z_i is to be determined by well-known processes.

3. Another course is open to us, leading to an auxiliary system of the same general character, but with a different number of F's, and consisting of a different number of equations. In some cases one process will be more convenient, in some cases the other; and, besides the advantage of having a choice, there is the further result that we get two distinct auxiliary systems, which must be equivalent, and thus may get hints as to the possible transformations of such systems.

For the application of the second method, we suppose that the equations to be solved are presented in what I have elsewhere* called the homogeneous form. The differential coefficients only occur in Jacobians of the n^{th} order; viz., in the determinants of the matrix

$$\left\| \begin{matrix} \frac{dz_1}{dx_1} & \cdots\cdots & \frac{dz_1}{dx_n} \\ \cdots & \cdots & \cdots \\ \frac{dz_m}{dx_1} & \cdots\cdots & \frac{dz_m}{dx_n} \end{matrix} \right\| \quad \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Each equation is algebraically homogeneous with respect to the Jacobians, and besides them involves only the independent variables $x_1 \dots x_n$. Should the given system not be homogeneous, it can be reduced to that form by transformation, the new system having m dependent, and, generally, $m+n$ independent variables. It will be convenient to keep m, n as the numbers of z's and x's in the transformed equations, and it appears that n must be greater than m.

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* "Proceedings of London Mathematical Society," Vol. ix., p. 41, *et sqq.*

The equations being in the homogeneous form, we may, instead of seeking the values of $$\frac{dz_i}{dx_j},$$ investigate the values of the Jacobians (1). To do this, it is only necessary to write down the relations which are identically satisfied by the Jacobians, and then, supposing these to be given by μ equations,

$$F_1 = 0, \;\; F_2 = 0 \;\ldots\; F_\mu = 0,$$

to transform these identities into their expressions in terms of F.

Since all the determinants of (1) involve $z_1 \ldots z_m$, any one of them will be distinguished by the subscripts of the x's it involves; we shall so indicate the Jacobians in the sequel, unless otherwise stated.

4. The first set of relations are purely algebraical, and arise from the fact that the Jacobians are determinants of a matrix. They may be thus expressed. Take any set of m suffixes, say, $1, 2 \ldots m$; and let $1', 2' \ldots m'$ represent groups, each of $m-1$ suffixes, and so arranged that each of $$11',\;\; 22' \ldots mm'$$ is composed of $1, 2 \ldots m$, and is reducible to this order by an even number of transpositions.* Let p be any subscript, and p' any group of $m-1$ subscripts, selected from $1, 2 \ldots m, m+1 \ldots n$. Then we have the identity $$11'.\,pp' = 1p'.\,p1' + 2p'.\,p2' + \ldots + mp'.\,pm' \;\ldots\ldots\ldots\ldots (2),$$ where pp' represents a Jacobian involving x's with the m subscripts pp', and similarly for the other symbols.

The number of independent equations (2) may be thus determined:—The Jacobians of the system (1), are in number $\dfrac{n\,!}{m\,!\;\overline{n-m}\,!}, = \mu$, say. We shall show that these μ Jacobians are, in virtue of (2), equivalent to a system of $m\,(n-m)+1$ only; the reduced system consisting of an arbitrarily selected Jacobian, say $(12 \ldots m)$, and the $m\,(n-m)$ formed therefrom by replacing one of the m symbols $1, 2 \ldots m$ by one of the $n-m$ symbols $m+1 \ldots n$. Hence the equations (2) are equivalent to

$$\mu - m\,(n-m) - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

independent relations.

It remains to show that the reduced system above described, is really equivalent to the complete system (1); that, the reduced system being known, all the determinants of the complete system are also

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* When $m = 2$, this arrangement is not possible, since $1'$, &c., are single subscripts. In this case the equation (2) is replaced by

$$12.\,pp' = 1p'.\,p2 - 2p'.\,p1$$

(or an equivalent), p, p' being any two subscripts.

The equation (2), and the special case just noted, are included under one form in a paper by Prof. Cayley, *Quart. Journ.*, xv., pp. 55—57.

known in virtue of (2). Suppose, then, we know the Jacobian $(12 \ldots m)$, and those derived from it by a single replacement (say, its *primary derivatives*). In (2), let p be any one of the $n-m$ symbols $m+1 \ldots n$, and suppose p' to be made up of one of these $n-m$ symbols, together with $m-2$ of the range $1, 2 \ldots m$. Then pp' is a *secondary derivative* of $(12 \ldots m)$, and, by properly choosing p, p', it may be made any one we please of the secondary derivatives. All the remaining symbols in (2) are primary derivatives (since each involves only one subscript outside the range $1, 2 \ldots m$), and are therefore known. Hence we can determine all the secondaries, when all the primary derivatives are known. Similarly, the tertiary derivatives can be found from the secondaries, and so on; so that, as stated above, all the determinants of the matrix (1) are known, if only we know the value of any one of them, and all its primary derivatives.

The relations (2) may be used in two ways. We may seek the value of each Jacobian of the complete set, and regard the equations (2) as so many data towards that end. Or we may use them at once to eliminate from a system of equations proposed for solution all the Jacobians, except an arbitrarily chosen one, and its primary derivatives. The latter appears to be the more convenient process, since we have not only a smaller number of Jacobians to determine, but the difficulties of notation are materially diminished. From the results of Art. 5, it will, however, be seen that the balance of convenience is in favour of retaining the complete set of Jacobians.

5. Besides the algebraical identities above discussed, there is another set, involving in their expression the differential coefficients of the Jacobians. They may be thus described:—Take any determinant of the matrix

$$\left\|\begin{matrix} \frac{d}{dx_1}, & \frac{d}{dx_2} & \cdots\cdots & \frac{d}{dx_n} \\ \frac{dz_1}{dx_1}, & \frac{dz_1}{dx_2} & \cdots\cdots & \frac{dz_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dz_m}{dx_1}, & \frac{dz_m}{dx_2} & \cdots\cdots & \frac{dz_m}{dx_n} \end{matrix}\right\| \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(4),$$

and develope it by the ordinary rule as a sum of products of the elements of the top line with their conjugate minors. For instance, supposing $m = 2$, we should have

$$\frac{d}{dx_i}(jk) - \frac{d}{dx_j}(ik) + \frac{d}{dx_k}(ij) = 0.$$

It is known* that these equations, combined with the algebraical equations of Art. 4, are necessary and sufficient to ensure that, for instance,

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* See "Proceedings of London Mathematical Society," Vol. x., pp. 55, *et sqq.*

$$(jk),\ (ik),\ (ij) = \frac{d\,(z_1,\ z_2)}{d\,(x_i,\ x_j,\ x_k)};$$

in other words, that there are functions z_1, z_2 such that (jk) is the Jacobian of them, with respect to x_j, x_k.

In the general case, we obtain

$$\frac{n\,!}{\overline{m+1}\,!\ \overline{n-m-1}\,!}$$

equations; viz., one equation is obtained for every group of $m+1$ columns, $i, j, k \ldots l$, that can be selected from the n columns of (4); the equation being

$$\frac{d}{dx_i}\,(jk\ldots l) - \frac{d}{dx_j}\,(ik\ldots l) + \frac{d}{dx_k}\,(ij\ldots l) - \&\text{c.} = 0 \quad \ldots\ldots(5).$$

These equations are mutually independent.

We can now choose one of the two alternatives suggested at the end of Art. 4. Suppose $n=4$, $m=2$. Then we have *one* algebraical relation

$$12\,.\,34 - 13\,.\,24 + 14\,.\,23 = 0 \ldots\ldots\ldots\ldots(a);$$

and *four* differential relations

$$\left.\begin{aligned}
\frac{d}{dx_2}\,.\,34 - \frac{d}{dx_3}\,.\,24 + \frac{d}{dx_4}\,.\,23 &= 0\\
\frac{d}{dx_1}\,.\,34 - \frac{d}{dx_3}\,.\,14 + \frac{d}{dx_4}\,.\,13 &= 0\\
\frac{d}{dx_1}\,.\,24 - \frac{d}{dx_2}\,.\,14 + \frac{d}{dx_4}\,.\,12 &= 0\\
\frac{d}{dx_1}\,.\,23 - \frac{d}{dx_2}\,.\,13 + \frac{d}{dx_3}\,.\,12 &= 0
\end{aligned}\right\} \ldots\ldots\ldots\ldots(b).$$

Say, we select 13 as principal Jacobian; then all the Jacobians except 24 are included in the *reduced system.* To eliminate this, we must replace it, in $(b1)$, $(b3)$, by its value deduced from (a). This being done, $(b1)$, after simplification, becomes an equation linear with respect to the differential coefficients of 34, 13, &c.; but the co-factors of these differential coefficients are binary products of 34, 13, &c. Also the four equations cannot be mutually dependent, for each contains a differential coefficient not contained in any of the others; viz., $\frac{d}{dx_3}\,13$, $\frac{d}{dx_4}\,13$, $\frac{d}{dx_1}\,13$, and $\frac{d}{dx_2}\,13$ occur, each in only one equation. Hence the effect of removing 24, is to increase the complexity of the differential equations, without diminishing their number. Besides this, it is not obvious what is the general form of the differential equations when only the Jacobians of a reduced system are explicitly involved; and the advantages of working with a smaller set of Jacobians seem to be more than counterbalanced by the disadvantages of losing such a simple general form as (5). Hence we elect to retain the full set of Jacobians, and to look upon (2) as given relations between them. The question is, it may be remarked, purely one of convenience, as the process of solution is virtually the same whichever course we adopt.

6. If the μ Jacobians $(12 \dots m)$, $(ij \dots l)$, &c., instead of being given explicitly in terms of $x_1 \dots x_n$, be given by means of μ equations

$$F_1 = 0, \ F_2 = 0 \ \dots \ F_\mu = 0,$$

the functions F must satisfy certain conditions in order that the resulting values of $(12 \dots m)$, &c., may satisfy the relations (5). It is easy to write these conditions down by making use of the following notation. The Jacobian of $F_1 \dots F_\mu$, with respect to the μ quantities $(12 \dots m)$, $(ij \dots l)$, &c., will be denoted by $\left| \ \right|$; the Jacobian formed from this by writing x_k instead of $(ij \dots l)$, will be written $\left| \begin{matrix} ij \dots l \\ k \end{matrix} \right|$; and similarly for others. Making use of the theorem

$$\left| \ \right| \times \frac{d}{dx_k}(ij \dots l) = \left| \begin{matrix} ij \dots l \\ k \end{matrix} \right|,$$

we obtain at once the following equivalent to (5),

$$\left| \begin{matrix} jk \dots l \\ i \end{matrix} \right| - \left| \begin{matrix} ik \dots l \\ j \end{matrix} \right| + \left| \begin{matrix} ij \dots l \\ k \end{matrix} \right| - \&c. = 0 \ \dots\dots\dots\dots\dots (6),$$

each term being, as explained above, a Jacobian of $F_1 \dots F_\mu$. The two systems (5), (6) are exactly equivalent; either is a consequence of the other.

7. The application of these results to the solution of a system of equations in the homogeneous form presents no difficulty. Suppose the system to consist of r equations

$$F_1 = 0, \ F_2 = 0, \ \dots \ F_r = 0.$$

We must supplement these by others

$$F_{r+1} = 0, \ \dots \ F_\mu = 0,$$

of which some are already known, being the equations (2). There are, therefore, only $m(n-m)+1-r$

functions, F, to be determined. To obtain these, we have only the datum that they must be such as to give values of $(12 \dots m)$, $(ij \dots l)$, &c., satisfying (5); in other words, we are only given that these unknown F's must satisfy the equations (6). These equations, as they stand, involve Jacobians of the μ^{th} order; but it is easy to make each of them lineal in Jacobians of the μ'^{th} order, μ' being the number of unknown F's; this is done by developing each Jacobian in (6), as a sum of products of conjugate minors, one of the minors in each term involving the μ' unknown F's, while the other contains all the known F's, and is therefore itself known.

8. Supposing the system (6) to be solved, then we can, from the μ equations
$$F = 0,$$
find $(12 \dots m)$, $(ij \dots l)$, &c., by the ordinary processes of algebra. It is, however, unnecessary to determine the whole system; for, as will appear immediately, the completion of the solution only requires a knowledge of what we have called a reduced system, consisting of one Jacobian and its primary derivatives. Say, we select $(12 \dots m)$ as the principal Jacobian, and represent it by $\left|\ \right|$; any prime derivative formed by replacing herein x_k by x_j, may be indicated by $\left|\begin{matrix} k \\ j \end{matrix}\right|$. Then, I say that $z_1, z_2 \dots z_m$ are the m solutions of the system

$$\left|\ \right| \frac{dz}{dx_j} - \left|\begin{matrix} 1 \\ j \end{matrix}\right| \frac{dz}{dx_1} - \left|\begin{matrix} 2 \\ j \end{matrix}\right| \frac{dz}{dx_2} - \dots - \left|\begin{matrix} m \\ j \end{matrix}\right| \frac{dz}{dx_m} = 0 \quad \dots\dots\dots(7),$$
$$j = m+1,\ m+2, \dots n.$$

The same thing is stated in other words, when we assert that $z_1, z_2 \dots z_m$ are m integrals of the system

$$\left|\ \right| dx_i + \left|\begin{matrix} i \\ m+1 \end{matrix}\right| dx_{m+1} + \left|\begin{matrix} i \\ m+2 \end{matrix}\right| dx_{m+2} + \dots + \left|\begin{matrix} i \\ n \end{matrix}\right| dx_n = 0 \dots(8),$$
$$i = 1, 2 \dots m.$$

The truth of (7) is immediately seen by observing that it is equivalent to the system

$$\left\| \begin{matrix} \frac{dz}{dx_1}, & \frac{dz}{dx_2} & \dots\dots & \frac{dz}{dx_n} \\ \frac{dz_1}{dx_1}, & \frac{dz_1}{dx_2} & \dots\dots & \frac{dz_1}{dx_n} \\ \dots & \dots & \dots & \dots \\ \frac{dz_m}{dx_1}, & \frac{dz_m}{dx_2} & \dots\dots & \frac{dz_m}{dx_n} \end{matrix} \right\| = 0,$$

which is evidently satisfied when z is any one of the functions $z_1, z_2 \dots z_m$, or any function of them; and is not satisfied for any other form of z.

In order that (7) or (8) may give m values for z, certain conditions must be satisfied by the coefficients $\left|\ \right|$, $\left|\begin{matrix} i \\ j \end{matrix}\right|$; and, supposing our reasoning to have been correct, these conditions must be satisfied identically in virtue of (2) and (5). It is easy to see that this is so:

in fact, the working has been given in a former paper of mine ("Proceedings of London Mathematical Society," Vol. x., pp. 66, 67): and thus we have an *a posteriori* verification of our results.

9. In the process of Art. 2, and in that developed in the succeeding articles of this paper, the solution of any system of equations involving

$$\frac{dz_\iota}{dx_j}$$

(say, for distinction, the *original* system) has been reduced to dependence upon the solution of a system of linear equations (say, the *auxiliary* system), concerning which some remarks will be made. The auxiliary system, in its most general form, may be described as consisting of ρ equations, each of which is linear in the Jacobians of the matrix

$$\frac{d\,(F_1, F_2 \dots F_\mu)}{d\,(\xi_1, \xi_2 \dots \xi_\nu)}, \quad (\nu > \mu).$$

Such equations are met with in the following cases:—

(A.) When the original system involves the first partial differential coefficients of one dependent variable with respect to n independent variables. In this case,

$$\mu = n, \;\; \nu = 2n+1, \;\; \rho = \tfrac{1}{2}n\,(n-1).$$

(B.) When the original system is of the kind discussed in the present paper. Adopting the method of Art. 2, we have

$$\mu = mn, \;\; \nu = mn+m+n, \;\; \rho = \tfrac{1}{2}mn\,(n-1).$$

(C.) When the second mode of solution is adopted for the systems of this paper, we are led to an auxiliary system, in which

$$\mu = \frac{n\,!}{m\,!\;\overline{n-m}\,!}, \text{ or } m\,(n-m)+1, \;\; \nu = \frac{n\,!}{m\,!\,\overline{n-m}\,!}+n, \;\; \rho = \frac{n\,!}{\overline{m+1}\,!\;\overline{n-m-1}\,!}.$$

When the larger value of μ is adopted, all the Jacobians in the auxiliary system belong to a reduced system; this is not the case when the smaller number is selected. The case (A) is the particular case of (B), (C) corresponding to $m=1$; and the values of μ, &c., found by putting $m=1$, agree with those given under (A), with one exception. In (C) it is necessary that the variables $z_1 \dots z_m$ should not appear in the auxiliary system. If, then, we reduce to the case of $m=1$, we find $\nu = 2n$, the variable z being excluded.

(D.) When the original system consists of partial differential equations with one dependent variable and n independent variables, and the order of the highest differential coefficient is r, we get a similar auxiliary system.* The number of F's is equal to the number of differential coefficients of order r; viz.,

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* "Proceedings of London Mathematical Society," Vol. ix., p. 89.

$$\mu = \frac{n+r-1!}{n-1!\ r!}.$$

The variables ξ are the differential coefficients of the r^{th} and all lower orders, together with z and the n independent variables. Hence

$$\nu = \frac{n+r!}{n!\ r!} + n.$$

The number of equations in the auxiliary system is most easily determined thus: If we differentiate each differential coefficient of the r^{th} order with respect to each of the n independent variables, we get

$$n \cdot \frac{n+r-1!}{n-1!\ r!}$$

different expressions. But the number of differential coefficients of

of order $r+1$ is only $\dfrac{n+r!}{n-1!\ r+1!}$.

Hence there must be $n \cdot \dfrac{n+r-1!}{n-1!\, r!} - \dfrac{n+r!}{n-1!\, r+1!}$

(differential) relations between the differential coefficients of order r. Each of these gives rise to one equation in the auxiliary system. Hence,

for this case, $\rho = \dfrac{n+r-1!\, r}{n-2!\, r+1!}$.

(E.) It is clear that the most general system of differential equations, viz., a system involving differential coefficients (of order r) of m dependent variables with respect to n independent variables, cannot be solved without an auxiliary system of type now considered. Indeed, the process of Art. 2 may be applied at once to such systems. Probably also a series of processes analogous to the second method of this paper would be applicable. It is not to be expected that the solution of the auxiliary system will suffice for the complete solution of the original system; indeed, in the case of $m=1$, we know that other equations must be solved as well as the auxiliary system now under consideration; but it is at least certain that the solution of this auxiliary system is an indispensable preliminary.

It is to be noticed that in every case mentioned the auxiliary system involves Jacobians of a reduced system only: an exception occurring in case (C) only, if we take the smaller value of μ.

10. The only case in which the auxiliary system has been solved is that marked (A) above. But the solution depends upon a certain transformation which is not possible unless certain relations obtain be-

tween μ, ρ. The auxiliary system is transformed* into another system, in which each equation connects two only of F's. Such an equation may be written $$(F_i, F_j) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$ On account of the symmetry of the auxiliary system with respect to F it is evident that the transformed system must involve all the equations that can be formed by giving i, j in (9) all values from λ to μ. There is nothing to prevent $$(F_i, F_i) = 0$$ being *identically* satisfied; but if any equation (9), in which i, j are different, should be satisfied independently of the form of F, then the same will be true of the whole system (9), and the transformation is illusory. Thus the transformed system consists of $\frac{1}{2}\mu(\mu-1)$ different equations; and accordingly ρ must be at least equal to $\frac{1}{2}\mu(\mu-1)$. It appears that another necessary condition for the transformation is that all the Jacobians involved must be the primary derivatives of a Jacobian which does not vanish. A reference to Art. 9 will show that these conditions are fulfilled only in the case (A).

Possibly the following process, which is at any rate general, may serve to give a solution in particular cases. It is of little use in the general case, because our process is stopped almost immediately by difficulties; but it may be that these disappear in certain particular instances; moreover the mere statement of the problem in a different form may not be without value.

Consider the equations as a system for the determination of one of the F's, say F_μ. Then each equation is a linear homogeneous equation of the first order, the coefficients of $\frac{dF_\mu}{dx_i}$ being a function (linear) of the Jacobians of $F_1 \ldots F_{\mu-1}$. In order that this system may have a common solution, certain solutions must be satisfied, the form of these being well known. These conditions serve as a system of equations for the determination of $F_1 \ldots F_{\mu-1}$, which, however, involve differential coefficients of the second order. The conditions that such a system, regarded as equations for $F_{\mu-1}$ say, should have a common solution, are not known; but they evidently will form a system connecting $F_1 \ldots F_{\mu-2}$. Proceeding in this way, we arrive ultimately at a system of equations involving only one F; or, it may be, at a system of identities each involving μ' F's, viz. $F_1 \ldots F_{\mu'}$. (The former case corresponds to $\mu' = 0$.) $F_1 \ldots F_{\mu'}$ are then perfectly arbitrary; and, leaving them thus arbitrary, we pass back through the successive systems above indicated, and from their solution determine $F_{\mu'+1} \ldots F_\mu$ in turn. There is the double difficulty of forming these systems and then solving them; but this does affect one result: viz., that the auxiliary system may be reduced to systems involving $\mu-1, \mu-2, \dots \mu'$ of the F's. These systems are not generally of the first order; though, as in case (A) of Art. 9, one of them may be so; and it seems not unlikely, bearing this case in mind, that a considerable simplification in the form of the subsidiary equations may be found when they can be written down.

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* This transformation and its converse are actually completed in Arts. 3, 4, 5 of a paper "On a General Method of solving Partial Differential Equations," *Proc. L. M. S.*, Vol. ix., pp. 76 *et sqq.*

The lack of symmetry in the above process is another objection to it; but it is not unprecedented.

Note on the Integral Solution of $x^2-2Py^2=-z^2$ *or* $\pm 2z^2$ *in certain Cases.* *By* SAMUEL ROBERTS, F.R.S.

[*Read March* 11*th*, 1880.]

1. I assume that P is a non-square number, the prime factors of which are of the form $8m+1$. The number is therefore uneven, and susceptible of the three forms $\alpha^2+16\beta^2$, $\alpha^2+8\beta^2$, and $\alpha^2-8\beta^2$.

The number of such representations is finite in the first two cases, and infinite in the last case.

The forms $\alpha^2+16\beta^2$ are comprised in

$$(8k\pm s)^2+16l^2,$$

where k, l remain unchanged in species throughout all the system for a given value of P, and s remains constant, having one of the values 1, 3.

In fact, if a^2+16b^2, $c^2+16d^2 \dots m^2+16n^2$ are representations of the prime factors of P (and these may be identical in groups, so that one or more prime factors enter in powers higher than the first), then, giving positive values to the letters, we may write

$$(a\pm i4b)(c\pm i4d)\dots(m\pm i4n) = \pm A \pm i4B \;\dots\dots\dots\dots (a)$$

with $P=A^2+16B^2$. If, therefore, $ac\dots m \equiv s \pmod{8}$, we shall have, independently of the ambiguity of the signs,

$$A=8k\pm s,\; B=l,$$

where k, l remain unchanged in species because the species of $\pm\mu\pm\nu\pm\pi\pm$ &c. is the same as that of $\mu+\nu+\pi+$ &c. Moreover, all the representations of P as the sum of two squares can be obtained from (a). The sign affecting s in two representations of the same number may be different.

It appears, similarly, that the representations of P by $\alpha^2+8\beta^2$ are comprised in

$$(8k\pm s)^2+8l^2,$$

where k, l remain unchanged in species in the same system, and s is constant.

With regard to the last form $\alpha^2-8\beta^2$, a new consideration enters, since the representations are infinite in number.

Suppose the prime factors of P are represented by a^2-8b^2, c^2-8d^2 ... m^2-8n^2, where we have $a^2>8b^2$, $c^2>8d^2$, &c., then all the similar representations of P immediately depending on these particular representations of the prime factors, are obtained by means of

$$(a\pm\sqrt{8}\,b)\,(c\pm\sqrt{8}\,d)\ldots(m\pm\sqrt{8}\,n)=\pm A\pm\sqrt{8}\,B\ldots\ldots(b).$$

Hence, if $ac\ldots m\equiv s$ (mod. 8),

$$A=8k+s,\ B=l,$$

where, as before, k, l remains unchanged in species, and s is constant, and remains of the same sign throughout. In this case, we suppose s to have one of the four values $+1$, -1, $+3$, -3.

In order to obtain all the other representations by means of the foregoing group derived from (b), it is necessary and sufficient to make use of

$$(A+\sqrt{8}\,B)\,(p\pm\sqrt{8}\,q)=\pm A'\pm\sqrt{8}\,B',$$

where $p^2-8q^2=1$.

For all the solutions of $x^2-8y^2=p$, where p is a prime, are obtained by means of

$$(x-\sqrt{8}\,y)\,(X\pm\sqrt{8}\,Y),$$

when $X^2-8Y^2=1$; consequently, when $x^2-8y^2=P'$, the prime factors of P', being of the form $\alpha^2-8\beta^2$, can be similarly obtained. As was noticed by Lagrange, and is now well known, this result does not hold always when P' is composite, because we have different conjugate pairs of incongruous roots satisfying $n^2-8\equiv 0$ (mod. P').

The representations of $p^2-8q^2=1$ are of the forms

$$(8\mu+1)^2-8\,(2\nu+1)^2,$$

or

$$(8\mu+3)^2-8\,(2\nu)^2.$$

So that we may write $A'=8k+s,\quad B'=l,$

or $A'=8k+3s,\ B'=l,$

the species k, l remaining unchanged, and s being constant and affected with the same sign in all the system.

Hence the representations of P by $\alpha^2-8\beta^2$ are comprised in

$$(8k+s)^2-8l^2,$$

or

$$(8k+3s)^2-8l^2.$$

The species of l is at once determined by means of $P\equiv 1 \pmod{16}$, or $P\equiv 9 \pmod{16}$, as the case may be.

2. We have next to determine certain relations in which the three classes of representations stand to one another, and, for this purpose, proceed by way of exclusion.

We cannot have

$$(8a\pm 1)^2+16\,(2\beta+1)^2=(8k\pm 1)^2+8\,(2l)^2 \dots\dots\dots\dots (c).$$

For suppose $8k\pm 1>8a\pm 1$, then

$$(8k\pm 1)^2-(8a\pm 1)^2=16\{(2\beta+1)^2-2l^2\}.$$

Any odd factor common to $(8k\pm 1)-(8a\pm 1)$ and $(8k\pm 1)+(8a\pm 1)$ is also common to $8k\pm 1$ and $8a\pm 1$. Also, $k\pm a$ must be odd, since

$$(k\pm a)\,\{4\,(k\mp a)\pm 1\}=(2\beta+1)^2-2l^2,$$

and since $4\,(k\mp a)\pm 1$ is of the form $8m\pm 3$, the first factor $k\pm a$ must have in common with the second a factor of the same form which will also be a factor common to $2\beta+1$ and l. It is also common to $8k\pm 1$ and $8a\pm 1$ by what precedes. This result is contrary to our primary supposition, and hence the equality (c) cannot hold.

A similar conclusion follows if we suppose $8a\pm 1>8k\pm 1$.

In the same way it will be found that the following equalities cannot hold

$$(8a\pm 1)^2+16\,(2\beta)^2 = (8k\pm 3)^2+8\,(2l+1)^2,$$
$$(8a\pm 3)^2+16\,(2\beta+1)^2 = (8k\pm 1)^2+8\,(2l+1)^2,$$
$$(8a\pm 3)^2+16\,(2\beta)^2 = (8k\pm 3)^2+8\,(2l)^2.$$

Again, we cannot have

$$(8a\pm 1)+16\,(2\beta+1)^2=(8k+1)^2-8\,(2l)^2,$$

or

$$(8k+3)^2-8\,(2l+1)^2.$$

In fact, taking the first form of the equality, and supposing

$$8k+1>8a\pm 1,$$

we have

$$(k\mp a)\,\{4\,(k\pm a)+1\}=(2\beta+1)^2+2l^2,$$

and $k\mp a=2n+1$ gives $4\,(k\pm a)+1=8m-3$. It follows that a factor of this form must be common to $8a\pm 1$ and $2\beta+1$, contrary to our supposition. The result is similar if we suppose $8a\pm 1>8k+1$ and, for like reasons, also the second form of the equality is impossible. Following the same method, we find that the following equalities cannot hold:—

$$(8a\pm 1)^2+16\,(2\beta)^2 = (8k-1)^2-8\,(2l)^2,$$

or

$$= (8k-3)^2-8\,(2l+1)^2;$$

$$(8a\pm 3)^2+16\,(2\beta+1)^2 = (8k+3)^2-8\,(2l)^2,$$

or

$$= (8k+1)^2-8\,(2l+1)^2;$$

$$(8a\pm 3)^2+16\,(2\beta)^2 = (8k-3)^2-8\,(2l)^2,$$

or

$$= (8k-1)^2-8\,(2l+1)^2.$$

On the other hand, we have the following compatible coexistent forms:

I.

$$(8\alpha\pm1)^2+16\,(2\beta+1)^2,\quad (8k\pm3)^2+8\,(2l+1)^2,\quad \begin{matrix}(8k-1)^2-8\,(2l)^2\\ (8k-3)^2-8\,(2l+1)^2.\end{matrix}$$

II.

$$(8\alpha\pm1)^2+16\,(2\beta)^2,\qquad (8k\pm1)^2+8\,(2l)^2,\qquad \begin{matrix}(8k+1)^2-8\,(2l)^2\\ (8k+3)^2-8\,(2l+1)^2.\end{matrix}$$

III.

$$(8\alpha\pm3)^2+16\,(2\beta+1)^2,\quad (8k\pm3)^2+8\,(2l)^2,\qquad \begin{matrix}(8k-1)^2-8\,(2l+1)^2\\ (8k-3)^2-8\,(2l)^2\quad .\end{matrix}$$

IV.

$$(8\alpha\pm3)^2+16\,(2\beta)^2,\qquad (8k\pm1)^2+8\,(2l+1)^2,\quad \begin{matrix}(8k+1)^2-8\,(2l+1)^2\\ (8k+3)^2-8\,(2l)^2\quad .\end{matrix}$$

3. We have next to apply the foregoing results to the equations $x^2-2Py^2=-z^2$ or $\pm 2z^2$, in which we shall suppose x to be prime to y.

In the first equation, let $y=p^2+4q^2$ or $y^2=(p^2-4q^2)^2+16p^2q^2$, $2y^2=(p^2-4q^2+4pq)^2+(p^2-4q^2-4pq)^2$, which last is of the form $(8\mu\pm1)^2+(8\nu\pm1)^2$, p being odd and q odd or even.

If, now, P belongs to set I., we get

$$2Py^2=(8u\pm3)^2+(8v\pm3)^2$$

by means of the product

$$\{8\alpha\pm1+i4\,(2\beta+1)\}\{8\mu\pm1+i\,(8\nu\pm1)\}.$$

Again, if y is of the form p^2+2q^2,

$$y^2=(p^2-2q^2)^2+8p^2q^2=(8\mu\pm1)^2+8\nu^2,$$

we have Py^2 of the form $(8u\pm3)^2+8u^2$ or $2Py^2=16u^2+2\,(8v\pm3)^2$.

Lastly, if $y=p^2-2q^2$, $y^2=(p^2+2q^2)^2-8p^2q^2=(8\mu+1)^2-8\,(2\nu)^2$, or $(8\mu+3)^2-8\,(2\nu+1)^2$, Py^2 is of the form $(8k-1)^2-8\,(2l)^2$ or $(8k-3)^2-8\,(2l+1)^2$, that is to say, of the form

$$2\,(8k-1\pm4l)^2-(8k-1\pm8l)^2$$

or

$$2\,\{8k-3\pm(4l+2)\}^2-\{8k-3\pm2\,(4l+2)\}^2,$$

and we get

$$2Py^2=4u^2-2\,(8v\pm1)^2.$$

Similar results are arrived at when P belongs to one of the three remaining sets; so that we get finally the following systems corresponding to the four sets:

I.

$$2Py^2=(8u\pm3)^2+(8v\pm3)^2,$$
$$2Py^2=16u^2+2\,(8v\pm3)^2,$$
$$2Py^2=4u^2-2\,(8v\pm1)^2.$$

II.

$$2Py^2=(8u\pm 1)^2+(8v\pm 1)^2,$$
$$2Py^2=16u^2+2\,(8v\pm 1)^2,$$
$$2Py^2=4u^2-2\,(8v\pm 1)^2.$$

III.

$$2Py^2=(8u\pm 1)^2+(8v\pm 1)^2,$$
$$2Py^2=16u^2+2\,(8v\pm 3)^2,$$
$$2Py^2=4u^2-2\,(8v\pm 3)^2.$$

IV.

$$2Py^2=(8u\pm 3)^2+(8v\pm 3)^2,$$
$$2Py^2=16u^2+2\,(8v\pm 1)^2,$$
$$2Py^2=4u^2-2\,(8v\pm 3)^2.$$

4. If P is a prime or the odd power of a prime, one of the three following equations must hold in integers: $x^2-2Py^2=-1$ or $+2$ or -2. And the forms of the last article shew that—

(i.) If P is of the set III., the equation $x^2-2Py^2=-1$ is always possible.

(ii.) If P is of the set I., the equation $x^2-2Py^2=+2$ is resoluble.

(iii.) If P is of the set IV., the equation $x^2-2Py^2=-2$ is resoluble.

Lastly, if P is of the set II., it remains doubtful which of the three forms of the equation is resoluble.

In stating negative results, we may regard P as having its wider meaning, and then we have the following:—

(i.) If P belongs to the set I., $x^2-2Py^2=-1$ or -2 is impossible in integers.

(ii.) If P belongs to the set III., $x^2-2Py^2=+2$ or -2 is impossible in integers.

(iii.) If P belongs to the set IV., $x^2-2Py^2=-1$ or $+2$ is impossible.

We get also impossible forms involving fourth powers from the consideration that $(8k-1)^2$, $(8k\pm 3)^2$ cannot be integer fourth powers.

Notes (1) *on a Geometrical Form of Landen's Theorem with regard to a Hyperbolic Arc;* (2) *On a Class of Closed Curves whose arcs possess the same property as two Fagnanian Arcs of an Ellipse.*

By JOHN GRIFFITHS, M.A., Jesus College, Oxford.

[*Read March* 11*th*, 1880.]

In the following notes the quantities r, p, t refer to a point on a curve, viz., r is the radius vector of the point, measured from the pole of the curve; p the length of the perpendicular from the pole upon the tangent at the point; and t the length intercepted on the tangent between its point of contact and the foot of the perpendicular in question; so that $r^2 = p^2 + t^2$.

1. If we take an ellipse (axes $2a, 2b$), the differential equation in r and p, when the centre is pole, is known to be $\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$; or, if $a = 1$, $b^2 = 1 - e^2$, this is $\frac{1-e^2}{p^2} = 2 - e^2 - r^2$. Hence, writing $1 - r^2 = \lambda^2$, $r^2 - (1-e^2) = \mu^2$, $2 - e^2 - r^2 = \nu^2$, we have

$$\begin{aligned} t^2 &= r^2 - p^2 \\ &= r^2 - \frac{1-e^2}{2-e^2-r^2} \\ &= \frac{(1-r^2)(r^2 - \overline{1-e^2})}{2-e^2-r^2} \\ &= \frac{\lambda^2\mu^2}{\nu^2}; \end{aligned}$$

i.e.,
$$t = \frac{\lambda\mu}{\nu}.$$

Again, if we consider another ellipse, semi-axes $(\overline{1+e}, \ \overline{1-e})$, we have $\frac{(1-e^2)^2}{p'^2} = 2(1+e^2) - r'^2$; or, since $\lambda^2 + \mu^2 = e^2$, $\mu^2 + \nu^2 = 1$,

this is
$$\frac{(\nu^2-\lambda^2)^2}{p'^2} = 4\mu^2 + 2\lambda^2 + 2\nu^2 - r'^2;$$

so that, if p' be taken equal $\nu - \lambda$, then $r'^2 = 4\mu^2 + (\nu-\lambda)^2$, and, consequently, $t' = 2\mu$.

Lastly, if we consider the hyperbola

$$\frac{e^2(1-e^2)}{p'^2} = 1 - 2e^2 + r''^2 \left(\text{transverse axis } 2e, \text{ eccentricity } \frac{1}{e}\right),$$

we have
$$\begin{aligned} \frac{(\lambda^2+\mu^2)(\nu^2-\lambda^2)}{p''^2} &= \mu^2 + \nu^2 - 2(\lambda^2+\mu^2) + r''^2 \\ &= \nu^2 - \mu^2 - 2\lambda^2 + r''^2; \end{aligned}$$

i.e., if p'' be taken $= \lambda$, then $r''^2 = \frac{\mu^2\nu^2}{\lambda^2} + \lambda^2$, and $t'' = \frac{\mu\nu}{\lambda}$.

Calling the two ellipses in question E, E', and the hyperbola H, we see, therefore, that there is a certain correspondence between a point P on E; P' on E'; and P'' on H; viz., if, for instance, P be an extremity of the minor axis of E, $\lambda = e$, $\mu = 0$, $\nu = 1$, and, therefore, $p' = 1 - e$, and $p'' = e$; *i.e.*, to an extremity of the minor axis of E corresponds an extremity of the minor axis of E', and, at the same time, an extremity of the transverse axis of H.

Landen's theorem follows at once, from the above, in the following geometrical form $2s'+s''-s = t'+t''$;

s being the arc of E measured from the end of the minor axis; s' that of E' measured in like manner from the corresponding point; s'' the arc of H measured from the end of the transverse axis; and t', t'' being the intercepts, as above, upon the tangents at P', P'' to the ellipse E' and hyperbola H, respectively.

Proof.—By an elementary formula $ds = \frac{rdr}{t}$; *i.e.*, $s = \int \frac{rdr}{t}$,

or
$$s = -\int \frac{\lambda d\lambda}{\frac{\lambda\mu}{\nu}}$$
$$= -\int \frac{\nu d\lambda}{\mu};$$

again,
$$s' = \int \frac{r'dr'}{t'}$$
$$= -\int \frac{2\lambda d\lambda + d(\lambda\nu)}{2\mu},$$

since $r'^2 = 4\mu^2+\nu^2+\lambda^2-2\lambda\nu$, and $\lambda^2+\mu^2 = e^2$, $\nu^2-\lambda^2 = 1-e^2$; and, consequently, $2r'dr' = -8\lambda d\lambda + 4\lambda d\lambda - 2d(\lambda\nu)$
$$= -4\lambda d\lambda - 2d(\lambda\nu).$$

Lastly,
$$s'' = \int \frac{r''dr''}{t''}$$
$$= \int \frac{p''dp''}{t''} + t'', \text{ since } r''^2 = p''^2 + t''^2$$
$$= \int \frac{\lambda d\lambda}{t''} + t'' = \int \frac{\nu d\nu}{\frac{\mu\nu}{\lambda}} + t'' = \int \frac{\lambda d\nu}{\mu} + t'';$$

therefore
$$2s'+s''-s = -\int \frac{2\lambda d\lambda}{\mu} - \int \frac{d(\lambda\nu)}{\mu} + \int \frac{\lambda d\nu}{\mu} + t'' + \int \frac{\nu d\lambda}{\mu}$$
$$= \int \frac{2\mu d\mu}{\mu} + t''$$
$$= 2\mu + t'',$$

or
$$2s'+s''-s = t'+t''.$$

2. Since the equation of the ellipse $\frac{a^2b^2}{p^2} = a^2+b^2-r^2$ remains unchanged when $a^2+b^2-p^2$ is written for r^2, and $a^2+b^2-r^2$ for p^2, we may say that this curve is self-corresponding, *i.e.*, to a point P (r, p) on it corresponds another point P' (r', p') such that $p'^2 = a^2+b^2-r^2$, and $r'^2 = a^2+b^2-p^2$; consequently $t' = t$.

Hence
$$ds = \frac{rdr}{t}$$
$$= -\frac{p'dp'}{t'}$$
$$= \frac{t'dt' - r'dr'}{t'}$$

or
$$ds = dt' + ds';$$

if s be measured from the extremity of the minor axis, say, and s' from that of the major axis; *i.e.*, $s - s' = t$, which is Fagnani's theorem.

It is easily seen, therefore, that in regard to this property the ellipse belongs to a class of closed curves given by the differential equation

$$F\{p^2(a^2+b^2-r^2),\ r^2(a^2+b^2-p^2),\ c\} = 0;$$

where F is an arbitrary function, and c is a constant.

The above result, as regards Landen's theorem, may be illustrated by means of figs. (1) and (2).

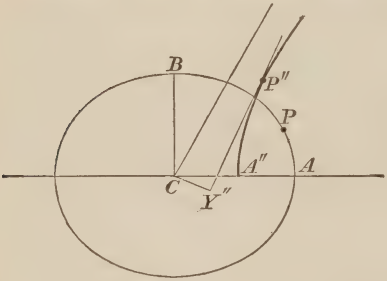

FIG. 1.

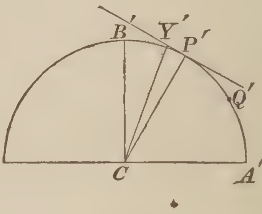

FIG. 2.

Where the semi-major axis of ellipse APB is $CA = 1$, and eccentricity e; semi-major axis of ellipse $A'P'B'$ is $CA' = 1+e$, and semi-minor axis $1-e$; semi-transverse axis of hyperbola $CA'' = e$, and eccentricity $\frac{1}{e}$.

Also P, P', P'' represent corresponding points on the respective curves. For example, the extremities of the respective minor axes B, B', and A'', a vertex of the hyperbola, simultaneously correspond; so that, if CY', CY'' be the perpendiculars from C on the respective tangents at P', P'', we have

$$P''Y'' - \text{arc } P''A'' = 2 \text{ arc } B'P' - \text{arc } BP - P'Y';$$

or, if Q' be the corresponding Fagnanian point of P', this may be written

$$P''Y'' - \text{arc } P''A'' = \text{arc } B'P' + \text{arc } A'Q' - \text{arc } BP.$$

It is to be observed that, when the point P'' goes off towards infinity, the point P on the ellipse E tends to coincide with A; and also the points P', Q' on the ellipse E' tend to coincide with each other; so that, in the limit, the difference between lengths of asymptote and infinite arc of hyperbola equals the difference between two elliptic quadrants—a result also due to Landen.

The above may be proved as follows:—When P'' is at infinity, $p''=0$; so that $\lambda=0$; *i.e.*, P coincides with A.

Also for P' we have $p'=\nu$, since $\lambda=0$, or $p'=\sqrt{1-e^2}$; but when P' and Q' coincide, we have
$$\begin{aligned}(p')^2 &= (1+e)(1-e)\\ &= 1-e^2.\end{aligned}$$

Hence P' and Q' coincide when P'' is at an infinite distance.

April 8th, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. James Barnard, M.A., of St. John's College, Cambridge, was elected a member, and Mr. T. Olver Harding was admitted into the Society.

The following papers were read:—

"A Form of the Equations determining the Foci and Directrices of a Conic whose Equation in Cartesian Coordinates is given:" Prof. Wolstenholme.

"The Application of Elliptic Coordinates, and Lagrange's Equations of Motion to Euler's Problem of Two Centres of Force:" Prof. Greenhill.

"Theorems in the Calculus of Operations:" Mr. J. J. Walker.

"On the Equilibrium of Cords and Beams in Certain Cases:" Mr. W. J. Curran Sharp.

"On Steady Motion and Vortex Motion in an Incompressible Viscous Fluid:" Mr. T. Craig.

"On Functions analogous to Laplace's Functions:" Mr. E. J. Routh, F.R.S.

The following presents were made to the Society:—

"Educational Times," April 1880.

"Washington Observations, 1876." Appendix II.—Reports on Telescopic Observations of the Transit of Mercury, May 5, 6, 1878.

"Atti della Reale Accademia dei Lincei,—Transunti," Fasc. 4°, Marzo 1880, Vol. iv.; Roma 1880.

Various papers and notes that have appeared in the "Quarterly Journal of Mathematics," and "The Messenger of Mathematics," during the year 1879: by J. W. L. Glaisher, F.R.S.

"Report of British Association Committee on Mathematical Tables," drawn up by J. W. L. Glaisher, F.R.S.

"On the Value of the Constant in Legendre's Formula for the number of Primes inferior to a given number:" by J. W. L. Glaisher, F.R.S. (Proceedings of the Cambridge Philosophical Society, Vol. iii., Part vii.)

"Note on the Enumerations of Primes of the forms $4n+1$ and $4n+3$" (British Association Report, 1879): by J. W. L. Glaisher, F.R.S.

"Separate Enumerations of Primes of the form $4n+1$ and of the form $4n+3$" (Proceedings of the Royal Society, No. 197, 1879): by J. W. L. Glaisher, F.R.S.

"On Definite Integrals involving Elliptic Functions" (Proceedings of Royal Society, No. 198, 1879): by J. W. L. Glaisher, F.R.S.

"Values of the Theta and Zeta Functions for Certain Values of the Argument" (Proceedings of Royal Society, No. 198, 1879): by J. W. L. Glaisher, F.R.S.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iii., Decembre 1879; Paris, 1879.

"Beiblätter zu den Annalen der Physik und Chemie," by G. and E. Wiedemann, Bd. iv., Stücke 4 & 5.

"Proceedings of Royal Society," Vol. xxx., No. 202.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. v., Part 1; Newhaven, 1880.

"On the Extra-meridian Determination of Time by means of a portable Transit-instrument," by Osmond Stone, A.M., Astronomer at the Cincinnati Observatory.

"American Journal of Mathematics, Pure and Applied," Vol. ii., No. 4, Dec. 1879; Baltimore, 1880.

On Functions analogous to Laplace's Functions.

By E. J. Routh, F.R.S.

[*Read April 8th*, 1880.]

Contents.

Arts. 47—49. Brief statement of some results given in a former paper on the subject. Object of the present paper.

Art. 50. Geometrical meaning of the symbol p.

Arts. 51, 52. The values of p are all real. They cannot be negative if the coefficients of the differential equation have certain specified signs.

Arts. 53—55. Some simple limits are found between which the values of p must lie. Foot-note on a theorem of Dr. Salmon's.

Arts. 56, 57. The function X, which corresponds to the least value of p, keeps one sign throughout.

Arts. 58—64. The roots of the equation $X=0$, which corresponds to any value of p, separates the roots of the equation $X=0$ which corresponds to a lesser value of p.

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47. In a paper which was published in the last volume of the Society's Transactions, I discussed a method of forming functions X, Y, &c., possessing the double property, that (1) $\int XY dx = 0$, and (2) that any function could be expanded in a series of such functions. This method was purely analytical, and depended on certain elementary properties of quadrics. The method, in the first instance, led us to certain equations of differences, and thence, by taking the limit, to certain differential equations. These when solved gave the functions X, Y, &c., all the constants, except one, being determined by given conditions at the limits. These equations might have any number of independent variables, and were all of an even order.

48. To save reference to the former paper, we may state that, when there is but one independent variable, the general form of the differential equation is

$$aX - \frac{d}{dx}\left(b\frac{dX}{dx}\right) + \frac{d^2}{dx^2}\left(c\frac{d^2X}{dx^2}\right) - \&c. = pA^2X,$$

where a, b, c, &c., and A^2 are all functions of x, to be chosen at our pleasure, but independent of p. The values of p and the constants of integration, except one, are determined by the conditions

$$\left.\begin{aligned} b\frac{dX}{dx} - \frac{d}{dx}\left(c\frac{d^2X}{dx^2}\right) + \&c. &= \lambda X \\ c\frac{d^2X}{dx^2} - \&c. \qquad\qquad &= \lambda'\frac{dX}{dx} \\ \&c. &= \&c. \end{aligned}\right\},$$

which hold at one limit, say $x = \alpha$, and

$$\left.\begin{aligned} b\frac{dX}{dx} - \frac{d}{dx}\left(c\frac{d^2X}{dx^2}\right) + \&c. &= -\mu X \\ \&c. &= \&c. \end{aligned}\right\},$$

which hold at the other limit, say $x = \beta$. Here $\lambda\lambda'$, &c., $\mu\mu'$, &c., are constants at our disposal, but they must be independent of p. In this way, we find

$$X = L\psi(x, p),$$

where L is an undetermined constant and p is given by an equation which we may write $f(p) = 0.$

If p, q be two different roots of this equation, then

$$\int_\alpha^\beta A\psi(x, p)\,\psi(x, q)\,dx = 0.$$

49. The simplest form of the equation will be

$$aX - \frac{d}{dx}\left(b\,\frac{dX}{dx}\right) = pA^2X,$$

where we restrict ourselves to the second order. The conditions at the limits are

$$\left.\begin{aligned} b\,\frac{dX}{dx} &= \lambda X, \quad \text{when } x = \alpha \\ b\,\frac{dX}{dx} &= -\mu X, \text{ when } x = \beta \end{aligned}\right\}.$$

In this form the equation represents the motion of heat in a bar. The functions a and b depend on the coefficients of cooling and conduction and on the area and perimeter of the section. With this interpretation, it is essential that the functions a, b, and A^2 should be positive from one end of the bar to the other. The constants λ and μ must also be positive.

This simple form of the equation has been discussed by C. Sturm in the first volume of Liouville's "Journal." I have not yet examined the whole paper, but he gives a very full introduction, in which he sums up his more remarkable results. These are of two kinds,—first, those which relate to the properties of the equation to find p, and secondly, those which relate to the properties of the function X.

(1). He remarks that Poisson has shown that the equation to find p has no imaginary roots. He now shows that the equation has no negative or equal roots.

(2). None of the functions X, Y, &c., can vanish without changing sign.

The first of these, *i.e.*, the function which corresponds to the least root of the equation to find p, preserves one sign from one end of the bar to the other. The second, *i.e.*, the function which corresponds to the next smallest root, changes sign once at a point somewhere between the two given extremities of the bar. The third changes sign twice, the fourth three times, and so on.

Two functions corresponding to two consecutive roots change sign alternately, one after the other, that which corresponds to the greater vanishing first, as we proceed from end of the bar to the other.

Analogous theorems are then proved relative to the maxima and minima values of these functions.

I propose to show in this paper that some of these results follow at once from elementary properties of the quadrics made use of in my former paper. Many of the properties are true in the general case, when we do not restrict the differential equation to be of the second order. They are also true when the functions are given by an equation of differences.

This, however, is not the communication alluded to in the preface to my former paper.

50. *To find the geometrical meaning of the symbol p.*

Consider the two quadrics

$$A_1^2 X_1^2 + A_2^2 X_2^2 + \ldots = H^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$2U = a_{11} X_1^2 + 2a_{12} X_1 X_2 + \ldots = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We seek (as in Art. 6 of the former paper) the common conjugate diameters of the two quadrics. Let $X_1 X_2 \ldots X_n$ be the coordinates of the extremity of one such diameter, then*

$$\left.\begin{aligned} \frac{dU}{dX_1} &= pA_1^2 X_1 \\ \frac{dU}{dX_2} &= pA_2^2 X_2 \\ \&\text{c.} &= \&\text{c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let this diameter cut the first quadric in P and the second in Q. This is, of course, a short mode of stating that, if $X_1 X_2 \ldots X_n$ be obtained from (1) and (3), then

$$OP^2 = X_1^2 + X_2^2 + \ldots + X_n^2;$$

but if $X_1 X_2 \ldots$ be obtained from (2) and (3), then

$$OQ^2 = X_1^2 + X_2^2 + \ldots + X_n^2.$$

If we now multiply equations (3) by $X_1 X_2 \ldots X_n$, and add the results, we find, if $X_1 X_2 \ldots X_n$ be the coordinates of Q,

$$1 = p\,(A_1^2 X_1^2 + \ldots + A_n^2 X_n^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

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* The property $\Sigma A^2 XY = 0$ follows at once from these equations without assuming the reality of the values of p, or referring to any geometrical property. Let q be another value of p, and let $Y_1 Y_2 \ldots$ be the corresponding values of $X_1 X_2 \ldots$, then we easily find $\quad X_1 \dfrac{dU}{dY_1} - Y_1 \dfrac{dU}{dX_1} = (p-q)\, A_1^2 X_1 Y_1.$

Substitute for U and sum for all the suffixes; we have

$$0 = (p-q)\, \Sigma A^2 XY.$$

This gives at once $$p = \frac{1}{H^2}\left(\frac{OP}{OQ}\right)^2 \quad\text{..................................(5).}$$

If we eliminate $X_1 X_2 \dots X_n$ from the equations (3), we of course get the determinant $$\begin{vmatrix} a_{11}-pA_1^2, & a_{12}, & \&c. \\ a_{12}, & a_{22}-pA_2^2, & \\ \&c. & & \end{vmatrix} = 0 \quad\text{.........................(6)}$$

already given in the article referred to in the former paper.

51. *The values of p given by the determinant are all real.*

Three different proofs of this proposition have been given by Dr. Salmon in his "Lessons on Higher Algebra." The proof given by Poisson, in Art. 90 of his "Théorie de la Chaleur," is applied to the case of a differential equation of the second order, but it may easily be extended to the general case of a differential equation or an equation of differences of any order.

If possible, let $p = h+k\sqrt{-1}$ be an imaginary root, then, substituting in equations (3), we have

$$\frac{X_1}{L_1+M_1\sqrt{-1}} = \frac{X_2}{L_2+M_2\sqrt{-1}} = \&c.$$

But we must also have another root, viz., $p = h-k\sqrt{-1}$. Substituting this also in the same equations and representing the new values of $X_1 X_2$, &c., by $Y_1 Y_2$, &c., we have

$$\frac{Y_1}{L_1-M_1\sqrt{-1}} = \frac{Y_2}{L_2-M_2\sqrt{-1}} = \&c.$$

But we also have $$A_1^2 X_1 Y_1 + A_2^2 X_2 Y_2 + \dots = 0.$$

This leads to $$A_1^2(L_1^2+M_1^2) + A_2^2(L_2^2+M_2^2) + \dots = 0,$$

which is impossible if $A_1^2 A_2^2 \dots$ be all positive or all negative.

Referring back to Art. 48 of this paper, *we infer that, if A^2 keeps one sign as x varies from one limit to the other, the equation $f(p) = 0$ has no imaginary roots.*

52. *To determine if any of the values of p can be negative.*

Since the values of p are real, it follows from equations (3) that the ratios of $X_1 X_2 \dots$ are all real.

In the quadric (1), the coefficients $A_1^2 A_2^2 \dots$ are all positive, so that, if there were but three coordinates, we should call (1) an ellipsoid. It easily follows that OP^2 is positive, and therefore OP is real.

In quadric (2), the coefficients $a_{11} a_{12} \dots$ are unrestricted, and may be either positive or negative, so that, if there were but three coordinates, we should say that (2) may be either an ellipsoid or a hyperboloid. Thus OQ^2 may be either positive or negative. In the former case p is positive, in the latter negative.

But if we wish to arrive at the differential equation

$$aX - \frac{d}{dx}\left(b\,\frac{dX}{dx}\right) + \frac{d^2}{dx^2}\left(\frac{d^2X}{dx^2}\right) - \&c. = pA^2X$$

in its most general form, we must, as in Art. 23 of the former paper, choose as our quadric (2)

$$2U = a_1X_1^2 + b_1(X_2 - X_1)^2 + c_1(X_3 - 2X_2 + X_1)^2 + \ldots$$
$$+ \text{similar terms with increased suffixes.}$$

If now a, b, c, as well as A^2, be positive from one limiting value of x to the other, then this quadric also is of the ellipsoid class, and thus all the values of p will be also positive.

This argument may, however, be even more generally applied. If we take the case of two independent variables as given in Art. 38 of the former paper, we see that the quadric $2U = 1$ is certainly of the ellipsoidal class, if we put the e's equal to zero. *So that, when the differential equation takes the form*

$$aX - \frac{d}{dx}\left(b\,\frac{dX}{dx}\right) - \frac{d}{dy}\left(c\,\frac{dX}{dy}\right) = pA^2X,$$

where a, b, c, A are positive over the area of integration, then the values of p are all positive. We may also apply the argument to the more general case in which the differential equation is of an order higher than the second.

53. Returning to the quadric (2), we see that some of the values of p may be negative, if one of the functions b, c, &c. becomes negative between the limits of integration. We may easily understand the geometrical meaning of this. If we put $A_1X_1 = \xi_1$, $A_2X_2 = \xi_2$, &c., the process described in Art. 50 is really that of making

$$\xi_1^2 + \xi_2^2 + \ldots = \text{max. or min.}$$

We are therefore finding what we may call the principal diameters of the quadric (2), when we treat $\xi_1\xi_2 \ldots$ as the running coordinates. By equation (4) of Art. 50, we see that p is the reciprocal of the square of any such semi-diameter. Thus, if p, q, r ... be the values of p given by the equation (6), the quadric (2) may be written

$$pA_1^2X_1^2 + qA_2^2X_2^2 + \ldots = 1.$$

Extending the phraseology of solid geometry, there will be no negative roots, one negative root, two negative roots, &c., according as the quadric (2) is an ellipsoid, a hyperboloid of one, two, &c. sheets.

54. We may also obtain some simple limits to the values of p. It is clear that the greatest semi-diameter must be greater than the greatest of the quantities found by putting all the ξ's, except one, equal to zero. Let us take for our quadric (2), the expression

$$\begin{aligned}2U = a_1X_1^2 &+ b_1(X_2-X_1)^2 \\ +a_2X_2^2 &+ b_2(X_3-X_2)^2 \\ +\&c.,&\end{aligned}$$

as given in Arts. 9 and 10 of the first paper. If we express the X's in terms of the ξ's we shall see that the least value of p must be less than the least of the quantities

$$\frac{a_1+\lambda}{A_1^2},\quad \frac{a_2+b_1+b_2}{A_2^2},\quad \frac{a_3+b_2+b_3}{A_3^2},\ \&c.\,;$$

where the λ of Art. 10 has been written separately from the a_1. In the same way, the greatest value of p must be greater than the greatest of these. In the limit, for a differential equation of the second order, *the least and greatest values of the function* $\frac{a+2b}{A^2}$ *lie between the least and greatest values of* p.

If $a+2b$ *become negative between the limits, some of the values of* p *must be negative.*

55. Taking the same quadric, and supposing the a's and b's to be positive, we notice that, whatever the ratios of $X_1X_2 \ldots X_n$ may be, the sum of their squares will be increased if we remove the terms containing b from the equation. Hence the greatest diameter of the quadric $2U=1$ is less than the greatest diameter of the quadric formed by omitting the terms containing the b's. *That is, the least value of* p *is greater than the least value of* $\frac{a}{A^2}$.*

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* Dr. Salmon remarks, at the beginning of his sixth lesson on *Higher Algebra*, that, if in the symmetrical determinantal equation

$$\Delta_1 = \begin{vmatrix} a_{11}-p, & a_{12}, & \&c. \\ a_{12}, & a_{22}-p, & \&c. \\ \&c. & \&c. & \&c. \end{vmatrix} = 0,$$

we suppress the first row and the first column, we have a second symmetrical determinantal equation which we may write $\Delta_2=0$ whose roots are all real, and separate those of the first determinantal equation. We may extend this convenient theorem as follows.

If we border the original determinant with *any* given quantities $l_1, l_2 \ldots$, and thus form a third equation, viz.,

$$\Delta_3 = \begin{vmatrix} a_{11}-p, & a_{12}, & \&c. & l_1 \\ a_{12}, & a_{22}-p, & \&c. & l_2 \\ \&c. & \&c. & & \&c. \\ l_1, & l_2, & \&c. & 0 \end{vmatrix} = 0,$$

56. *To show that, if the functions a, b (and of course A^2) be positive from one limit of integration to the other, then the function X, which corresponds to the least root of the equation to find p, keeps one sign from one limit to the other.*

As before, let $X_1 X_2 \ldots X_n$ be the successive values of the function, and let the quadric (2) be

$$\begin{aligned} 2U &= a_1 X_1^2 + b_1 (X_2 - X_1)^2 \\ &\quad + \text{same with increased suffixes} \\ &= 1. \end{aligned}$$

Let us suppose that p has its least value, then, by equation (4), Art. 50, we have to make

$$R^2 = A_1^2 X_1^2 + A_2^2 X_2^2 + \ldots + A_n^2 X_n^2$$

as great as possible.

If possible, let any of the X's, say $X_3 X_5 \ldots$, have a different sign from the rest. Then, by changing the signs of these, leaving their numerical values unaltered, we clearly decrease the value of U. Since $2U$ is now less than unity, and $X_1 X_2 \ldots X_n$ all have now the same sign, we can numerically increase all of them by the same quantity, so as to make $2U$ equal to unity. It follows that R has been increased. Hence R is not the greatest possible unless $X_1 X_2 \ldots X_n$ all have the same sign.

57. This theorem is also true when the function X is obtained from the differential equation of an order higher than the second. The

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then the roots of this will also be all real and will separate the roots of the first determinantal equation. If we make all the l's zero except one, we get the proposition as enumerated by Dr. Salmon.

To prove this we notice that, if $\alpha_{11} \alpha_{22} \ldots$ be the minors of the constituents in the leading diagonal of Δ_1, then the third determinant (by Art. 37 of Dr. Salmon's *Algebra*) may be written in the form

$$\Delta_3 = \Delta'_3 = -\left\{ l_1 \sqrt{\alpha_{11}} + l_2 \sqrt{\alpha_{22}} + \&c. \right\}^2$$

whenever p has such a value that $\Delta_1 = 0$. Suppose p to take in succession the values of the roots of $\Delta_1 = 0$, beginning at the least. Then, at each change in the value of p, the minors $\alpha_{11} \alpha_{22} \ldots$ will change sign, because each is of the form Δ_2. Hence Δ'_3 will change sign also. Thus a root of the equation $\Delta_3 = 0$ lies between each adjacent two of the roots of $\Delta_1 = 0$.

This proposition may be stated in geometrical language. The roots of the determinantal equation $\Delta_1 = 0$ are the squares of the reciprocals of the principal semi-diameters of the quadric (2) in Art. 50, and what we have proved amounts to this,—the lengths of the principal diameters of any section, say,

$$l_1 X_1 + l_2 X_2 + \ldots = 0,$$

separate the lengths of those of the original quadric. In this form the proposition admits of a very easy proof by referring the quadric to its principal axes.

In the same way, if we border the determinant Δ_3 with a second set of arbitrary quantities, with zero in the corner, we get a new determinantal equation $\Delta_4 = 0$ whose roots are all real, and separate those of $\Delta_3 = 0$.

proof, however, is not of the same simple character as when we restrict the quadric to the form given in this article. I have, therefore, not thought it advantageous to give the proof at length.

58. When the function X is given by an equation of differences or a differential equation of the second order, the properties mentioned in Art. 49 are simple corollaries from the following lemma.

Consider the equation of differences

$$a_x X_x - \Delta\,(b_{x-1}\Delta X_{x-1}) = pA_x^2\, X_x.$$

If q be another value of p, we have

$$a_x Y_x - \Delta\,(b_{x-1}\Delta Y_{x-1}) = qA_x^2\, Y_x.$$

Eliminating the function a_x, we find

$$(q-p)\, A_x^2\, X_x Y_x = b_x\,(X_{x+1} Y_x - X_x Y_{x+1}) - b_{x-1}\,(X_x Y_{x-1} - X_{x-1} Y_x).$$

This gives

$$(q-p)\,[A_m^2\, X_m Y_m + \ldots + A_x^2\, X_x Y_x] = b_x\,(X_{x+1} Y_x - X_x Y_{x+1})$$
$$- b_{m-1}\,(X_m Y_{\ -1} - X_{m-1} Y_m).$$

The right-hand side may also be written

$$b_x\,(Y_x \Delta X_x - X_x \Delta Y_x) - b_{m-1}\,(Y_{m-1}\Delta X_{m-1} - X_{m-1}\Delta Y_{m-1}).$$

59. Cor. 1.—Consider the full series of values $X_1 X_2 \ldots X_n$ arranged in order. We shall have ranges of positive and negative values succeeding each other. Let $X_m \ldots X_x$ be one of these ranges in which all the constituents have one sign, while those on each side, viz., X_{m-1} and X_{x+1}, have the opposite sign. *We shall prove that, if $q > p$, there is one change of sign at least in the corresponding range of Y's extending from Y_{m-1} to Y_{x+1} both inclusive.*

For, if possible, let all these Y's have one sign, then every one of the four terms on the right-hand side of the equality in the lemma has the sign opposite to that of the product $X_x Y_x$. Hence the lemma could not be true.

We have here made no assumption as to the function a_x, but b_x and A_x^2 have been supposed to have the same sign from one limit to the other.

60. Cor. 2.—Consider next a double range of values, say $X_l \ldots X_m \ldots X_x$, such that all the constituents from X_l to X_{m-1} have one sign, say, negative, and $X_m \ldots X_x$ have the other sign, while X_{l-1} and X_{x+1} have opposite signs to the adjacent constituents in order to make the double range complete. Then, by Cor. 1, if $q > p$, Y must change sign between Y_{l-1} and Y_l, and also between Y_{l-1} to Y_{x+1}. *We shall now prove that a single change of sign between Y_{l-1} and Y_l will not suffice for both these requirements.*

For, if it did, the products $X_l Y_l \dots X_x Y_x$ would all have the same sign. But, writing l for m in the equation of the lemma, every one of the four terms on the right-hand side has the sign opposite to that of the products $X_x Y_x$, and thus again the lemma could not be true.

In the same way, if we consider a complete triple range of values, say, $X_k \dots X_l \dots X_m \dots X_x$, so that X changes sign twice as x varies from one limit to the other; then, by Cor. (1), Y must change sign between X_{k-1} and X_{l+1}, X_{l-1} and X_{m+1}, X_{m-1} and X_{x+1}. But it follows exactly as before that two changes of sign will not suffice for all three requirements.

61. Cor. 3.—Consider the range of values $X_1 \dots X_x$ all of one sign, beginning at one extremity of the full series, and such that X_{x+1} has the opposite sign. *We shall prove that, if $q > p$, there is one change of sign at least in the corresponding range of Y's extending from Y_1 to Y_{x+1}.*

In this case the range begins at one extremity; we have therefore the conditions $b_0(X_1 - X_0) = \lambda X_1$, $b_0(Y_1 - Y_0) = \lambda Y_1$,

which hold at that extremity. The equality in the lemma becomes therefore

$$(q-p)(A_1 X_1 Y_1 + \dots A_x X_x Y_x) = b_x(X_{x+1} Y_x - X_x Y_{x+1}).$$

If, then, all the Y's from Y_1 to Y_{x+1} had the same sign, every term on the left-hand side would have the same sign, and every term on the right-hand side would have the opposite sign, and thus the equality could not exist.

Similar remarks apply to a range terminating at the other extremity.

62. Cor. 4.—Lastly, consider all the n series

$$X_1 X_2 \dots X_n,$$
$$Y_1 Y_2 \dots Y_n,$$
&c.,

corresponding to the n values of p, arranged in order of magnitude, beginning at the least.

By the preceding corollaries, each of these series must have at least one more change of sign than any series above it. As there are but n terms in each series, the lowest, or n^{th}, can have only $n-1$ changes of sign.

Hence the first series has no change of sign, the second has only one change, the third has only two, and so on. Also, the changes of sign in each series alternate, in the manner already explained, with the changes of sign in any series above it.

63. It should be noticed that, in Cor. 1 and 2, no use has been made

of the conditions at the limits. In these propositions, therefore, p and q are any arbitrary quantities, except that q must be greater than p.

In Cor. 3, the conditions at one limit are introduced, so that the results of these three corollaries are true, if only $\frac{X_1}{X_0} = \frac{Y_1}{Y_0}$ at one limit.

Finally, in Cor. 4, the conditions at both limits are assumed to be satisfied, and therefore p and q must now be different roots of the equation

$$f(p) = 0.$$

64. If we take the second form of the lemma given in Art. 58, we may prove that similar theorems hold regarding the successive maxima and minima of the functions X and Y. But the mode of proof is very similar to that just used. There does not appear to be sufficient novelty to render it necessary to lengthen this paper by repeating the arguments with the necessary variations.

A Form of the Equations determining the Foci and Directrices of a Conic whose Equation in Cartesian Coordinates is given. By Prof. Wolstenholme.

[*Read April 8th*, 1880.]

If the rational equation of a conic referred to coordinate axes inclined at an angle ω, be $u = 0$, let (x, y) be the coordinates of a focus, and (X, Y) current coordinates. Then, if we move the origin to the focus, the equation will become

$$u + X\frac{du}{dx} + Y\frac{du}{dy} + \tfrac{1}{2}\left(X^2\frac{d^2u}{dx^2} + Y^2\frac{d^2u}{dy^2} + 2XY\frac{d^2u}{dx\,dy}\right) = 0.$$

But, the origin being a focus, the equation must be of the form

$$X^2 + Y^2 + 2XY\cos\omega = (pX + qY + r)^2.$$

Hence $$\frac{p^2-1}{\dfrac{d^2u}{dx^2}} = \frac{q^2-1}{\dfrac{d^2u}{dy^2}} = \frac{r^2}{2u} = \frac{qr}{\dfrac{du}{dy}} = \frac{rp}{\dfrac{du}{dx}} = \frac{pq-\cos\omega}{\dfrac{d^2u}{dx\,dy}} \; (\equiv \lambda).$$

We can deduce the equations

$$\left(\lambda\frac{d^2u}{dx^2} + 1\right)2u = \lambda\left(\frac{du}{dx}\right)^2, \quad \left(\lambda\frac{d^2u}{dy^2} + 1\right)2u = \lambda\left(\frac{du}{dy}\right)^2,$$

$$\left(\lambda\frac{d^2u}{dx\,dy} + \cos\omega\right)2u = \lambda\frac{du}{dx}\frac{du}{dy};$$

or,
$$\lambda\frac{d}{dx}\left(\frac{\frac{du}{dx}}{\sqrt{u}}\right)=\frac{\lambda}{2u^{\frac{3}{2}}}\left\{2u\frac{d^2u}{dx^2}-\left(\frac{du}{dx}\right)^2\right\}=-\frac{1}{\sqrt{u}},$$

$$\lambda\frac{d}{dy}\left(\frac{\frac{du}{dy}}{\sqrt{u}}\right)=-\frac{1}{\sqrt{u}},\quad \lambda\frac{d}{dy}\left(\frac{\frac{du}{dx}}{\sqrt{u}}\right)=-\frac{1}{\sqrt{u}}\cos\omega.$$

If we write u^2 in place of u, we obtain the very compact forms

$$\frac{d^2u}{dx^2}=\frac{d^2u}{dy^2}=\frac{d^2u}{dx\,dy}\sec\omega\left(=-\frac{1}{2\lambda u}\right)$$

for the equations determining the foci. I do not remember ever to have seen the equations given in these forms, which perhaps deserve record for their remarkable compactness.

The equation of the directrix, referred to the original axes, will be

$$(X-x)\,p+(Y-y)\,q+r=0,$$

or
$$\frac{X-x}{qr}+\frac{Y-y}{rp}+\frac{1}{pq}=0.$$

But $qr=\lambda\frac{du}{dy}$, $rp=\lambda\frac{du}{dx}$, $pq=\lambda\frac{d^2u}{dx\,dy}+\cos\omega=\lambda\frac{\frac{du}{dx}\frac{du}{dy}}{2u}$, whence the equation of the directrix will be (when $u=0$ is the equation of the curve)
$$(X-x)\frac{du}{dx}+(Y-y)\frac{du}{dy}+2u=0;$$

or (when $u^2=0$ is the equation of the curve)

$$(X-x)\frac{du}{dx}+(Y-y)\frac{du}{dy}+u=0,$$

where x, y are to be found from the equations

$$\frac{d^2u}{dx^2}=\frac{d^2u}{dy^2}=\frac{d^2u}{dx\,dy}\sec\omega.$$

Of course this equation might be written down at once as that of the polar of the focus.

[As an example, suppose

$$u^2=\frac{x^2}{8}+\frac{y^2}{5}-1,\text{ and }\omega=\frac{\pi}{3},$$

$$\frac{du}{dx}=\frac{x}{8u},\quad \frac{du}{dy}=\frac{y}{5u},\quad \frac{d^2u}{dx\,dy}=-\frac{xy}{40u^3},$$

$$\frac{d^2u}{dx^2}=\frac{\frac{y^2}{5}-1}{8u^3},\quad \frac{d^2u}{dy^2}=\frac{\frac{x^2}{8}-1}{5u^3},$$

and the equations for the foci are

$$y^2-5 = x^2-8 = -\,xy\sec\omega = -\,2xy;$$

therefore $\dfrac{y^2+2xy}{5} = \dfrac{x^2+2xy}{8}$, $5x^2-6xy-8y^2=0$, $x=2y$, or $-\dfrac{4y}{5}$,

$$x=2y \text{ gives } y^2=1,\quad x=-\frac{4y}{5} \text{ gives } y^2=-\frac{25}{3};$$

thus the real foci are (2, 1), (−2, −1); and the impossible

$$\left(\pm\frac{4}{\sqrt{-3}},\ \mp\frac{5}{\sqrt{-3}}\right);$$

while the real directrices are $\dfrac{x}{4}+\dfrac{y}{5}=\pm 1$; and the impossible directrices $\dfrac{x}{2}-y=\pm\sqrt{-3}$.]

The Application of Elliptic Coordinates and Lagrange's Equations of Motion to Euler's Problem of Two Centres of Force. By A. G. GREENHILL, M.A.

[*Read April 8th*, 1880.]

Denoting by $2c$ the distance between the centres of force; then, if $c\cos\theta$, $c\sin\theta$ be the $\frac{1}{2}$ axes of the hyperbola $c\cosh\phi$, $c\sinh\phi$ of the ellipse, passing through a point, and having their foci at the centres of force, θ and ϕ may be called the elliptic coordinates of the point; and if the axes of these conics be taken as coordinate axes, then the Cartesian coordinates of the point are $x=c\cos\theta\cosh\phi$ and $y=c\sin\theta\sinh\phi$.

Therefore, for a particle of unit mass, the kinetic energy

$$\begin{aligned}
T &= \tfrac{1}{2}(\dot{x}^2+\dot{y}^2)\\
&= \tfrac{1}{2}c^2\{(-\sin\theta\cosh\phi\,\dot{\theta}+\cos\theta\sinh\phi\,\dot{\phi})^2\\
&\qquad+(\cos\theta\sinh\phi\,\dot{\theta}+\sin\theta\cosh\phi\,\dot{\phi})^2\}\\
&= \tfrac{1}{2}c^2\{(\sin^2\theta\cosh^2\phi+\cos^2\theta\sinh^2\phi)\dot{\theta}^2\\
&\qquad+(\cos^2\theta\sinh^2\phi+\sin^2\theta\cosh^2\phi)\dot{\phi}^2\}\\
&= \tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)(\dot{\theta}^2+\dot{\phi}^2).
\end{aligned}$$

If r, s denote the distances of the particle from the centres of force, then

$$r = c(\cosh\phi-\cos\theta),$$
$$s = c(\cosh\phi+\cos\theta);$$

and if A, B be the strengths of the centres of force, supposed to attract with intensity inversely proportional to the square of the distance, the gravitation potential

$$U = \frac{A}{r} + \frac{B}{s}$$
$$= \frac{A}{c\,(\cosh\phi - \cos\theta)} + \frac{B}{c\,(\cosh\phi + \cos\theta)}.$$

[If we have a third centre of force, midway between the other two, attracting with intensity proportional to the distance, of strength C, we must add to this value of U the term

$$\tfrac{1}{2}C\,(x^2+y^2)$$
$$= \tfrac{1}{2}Cc^2\,(\cos^2\theta\cosh^2\phi + \sin^2\theta\sinh^2\phi)$$
$$= \tfrac{1}{4}Cc^2\,(\cosh 2\phi + \cos 2\theta).]$$

Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot\theta}\right) - \frac{\delta T}{\delta\theta} = \frac{\delta U}{\delta\theta},$$
$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot\phi}\right) - \frac{\delta T}{\delta\phi} = \frac{\delta U}{\delta\phi};$$

δ denoting partial, and d total differentiation.

Therefore $\tfrac{1}{2}c^2\,(\cosh 2\phi - \cos 2\theta)\,\ddot\theta + \tfrac{1}{2}c^2\sin 2\theta\,(\dot\theta^2 - \dot\phi^2) + \sinh 2\phi\;\dot\theta\dot\phi$

$$= -\frac{A}{c}\,\frac{\sin\theta}{(\cosh\phi - \cos\theta)^2} + \frac{B}{c}\,\frac{\sin\theta}{(\cosh\phi + \cos\theta)^2}$$
$$- \tfrac{1}{2}Cc^2\sin 2\theta \;\ldots\ldots\ldots\; (1),$$

and $\tfrac{1}{2}c^2\,(\cosh 2\phi - \cos 2\theta)\,\ddot\phi + \sin 2\theta\,\dot\theta\dot\phi - \tfrac{1}{2}c^2\sinh 2\phi\,(\dot\theta^2 - \dot\phi^2)$

$$= -\frac{A}{c}\,\frac{\sinh\phi}{(\cosh\phi - \cos\theta)^2} - \frac{B}{c}\,\frac{\sinh\phi}{(\cosh\phi + \cos\theta)^2}$$
$$+ \tfrac{1}{2}Cc^2\sinh 2\phi \;\ldots\ldots\ldots\; (2).$$

If we multiply (1) by $\dot\theta$, and (2) by $\dot\phi$, add and integrate, we obtain

$$\tfrac{1}{4}c^2\,(\cosh 2\phi - \cos 2\theta)\,(\dot\theta^2 + \dot\phi^2)$$
$$= \frac{A}{c}\,\frac{1}{\cosh\phi - \cos\theta} + \frac{B}{c}\,\frac{1}{\cosh\phi + \cos\theta}$$
$$+ \tfrac{1}{4}Cc^2\,(\cosh 2\phi + \cos 2\theta) - H \;\ldots\ldots\ldots\; (3),$$

the equation of energy.

To obtain the second integral of these equations of motion, multiply

(1) by $\cosh 2\phi\,\dot\theta$, and (2) by $\cos 2\theta\,\dot\phi$, and add; then

$$\tfrac{1}{4}c^2\frac{d}{dt}(\cosh 2\phi-\cos 2\theta)(\cosh 2\phi\,\dot\theta^2+\cos 2\theta\,\dot\phi^2)$$

$$=-\frac{A}{c}\,\frac{\sin\theta\cosh 2\phi\,\dot\theta+\cos 2\theta\sinh\phi\,\dot\phi}{(\cosh\phi-\cos\theta)^2}$$

$$+\frac{B}{c}\,\frac{\sin\theta\cosh\phi\,\dot\theta-\cos 2\theta\sinh\phi\,\dot\phi}{(\cosh\phi+\cos\theta)^2}$$

$$-\tfrac{1}{2}Cc^2(\sin 2\theta\cosh 2\phi\,\dot\theta-\cos 2\theta\sinh 2\phi\,\dot\phi);$$

and, integrating,

$$\tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)(\cosh 2\phi\,\dot\theta^2+\cos 2\theta\,\dot\phi^2)$$

$$=\frac{A}{c}\,\frac{2\cos\theta\cosh\phi-1}{\cosh\phi-\cos\theta}-\frac{B}{c}\,\frac{2\cos\theta\cosh\phi+1}{\cosh\phi+\cos\theta}$$

$$+\tfrac{1}{4}Cc^2\cos 2\theta\cosh 2\phi-D\;\ldots\ldots\ldots\;(4),$$

equivalent to Euler's second integral, D and H being arbitrary constants, determined by the initial circumstances of the motion.

From (3) and (4),

$$\tfrac{1}{4}c^2(\cosh 2\phi-\cos 2\theta)^2\,\dot\theta^2$$

$$=\frac{A}{c}\,\frac{2\cos\theta\cosh\phi-1-\cos 2\theta}{\cosh\phi-\cos\theta}-\frac{B}{c}\,\frac{2\cos\theta\cosh\phi+1+\cos 2\theta}{\cosh\phi+\cos\theta}$$

$$-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta$$

$$=2\,\frac{A-B}{c}\cos\theta-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta\;\ldots\ldots\ldots\ldots\ldots\ldots\;(5),$$

$$\tfrac{1}{4}Cc^2(\cosh 2\phi-\cos 2\theta)^2\,\dot\phi^2$$

$$=\frac{A}{c}\,\frac{\cosh 2\phi+1-2\cos\theta\cosh\phi}{\cosh\phi-\cos\theta}+\frac{B}{c}\,\frac{\cosh 2\phi+1+2\cos\theta\cosh\phi}{\cosh\phi+\cos\theta}$$

$$+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi$$

$$=2\,\frac{A+B}{c}\cosh\phi+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi\;\ldots\ldots\ldots\ldots\ldots\;(6).$$

Therefore

$$\left(\frac{d\theta}{d\phi}\right)^2=\frac{2\,\dfrac{A-B}{c}\cos\theta-\tfrac{1}{4}Cc^2\cos^2 2\theta-D+H\cos 2\theta}{2\,\dfrac{A+B}{c}\cosh\phi+\tfrac{1}{4}Cc^2\cosh^2 2\phi+D-H\cosh 2\phi},$$

the differential equation of the orbit, a differential equation in which the variables θ and ϕ are separated.

Euler employs new variables u and v, such that $u=\tan\tfrac{1}{2}\theta$, $v=\tanh\tfrac{1}{2}\phi$; his p and q being respectively $c\cos\theta$ and $c\cosh\phi$.

Then $\dfrac{d\theta}{du} = \dfrac{2}{1+u^2}$, $\cos\theta = \dfrac{1-u^2}{1+u^2}$, $\cos 2\theta = 2\left(\dfrac{1-u^2}{1+u^2}\right)^2 - 1$,

$$\frac{d\phi}{dv} = \frac{2}{1-v^2},\quad \cosh\phi = \frac{1+v^2}{1-v^2},\quad \cosh 2\phi = 2\left(\frac{1+v^2}{1-v^2}\right)^2 - 1\,;$$

and if C be put $= 0$, the differential equation becomes

$$\frac{du^2}{2\dfrac{A-B}{c}(1-u^4) - D(1+u^2)^2 + H(1-u^2)^2}$$

$$= \frac{dv^2}{2\dfrac{A+B}{c}(1-v^4) + D(1-v^2)^2 - H(1+v^2)^2} = d\lambda^2, \text{ suppose};$$

and therefore u and v are elliptic functions of λ, and, by the elimination of λ, we obtain the equation of the orbit in terms of u and v, or θ and ϕ.

The integral may also be written

$$\int \frac{du}{\sqrt{\left\{H-D+2\dfrac{A-B}{c} - 2(H+D)u^2 + \left(H-D-2\dfrac{A-B}{c}\right)u^4\right\}}}$$

$$\pm \int \frac{dv}{\sqrt{\left\{D-H+2\dfrac{A+B}{c} - 2(D+H)v^2 + \left(D-H-2\dfrac{A+B}{c}\right)v^4\right\}}} = E,$$

a constant;

and, using the notation

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \text{arg sn}\,(x, k),$$

this equation may be written

$$\frac{1}{\sqrt{\left[H+D+2\sqrt{\left\{HD+\left(\dfrac{A-B}{c}\right)^2\right\}}\right]}}$$

$$\text{arg sn}\left\{\frac{H-D-2\dfrac{A-B}{c}}{H+D-2\sqrt{\left\{HD+\left(\dfrac{A-B}{c}\right)^2\right\}}}\,u,\ k_1\right\}$$

$$\pm \frac{1}{\sqrt{\left[D+H+2\sqrt{\left\{DH+\left(\dfrac{A+B}{c}\right)^2\right\}}\right]}}$$

$$\text{arg sn}\left\{\frac{D-H-2\dfrac{A+B}{c}}{D+H-2\sqrt{\left\{DH+\left(\dfrac{A+B}{c}\right)^2\right\}}}\,v,\ k_2\right\} = E,$$

when
$$k_1 = \frac{H+D-2\sqrt{\left\{HD+\left(\frac{A-B}{c}\right)^2\right\}}}{H+D+2\sqrt{\left\{HD+\left(\frac{A-B}{c}\right)^2\right\}}},$$
$$k_2 = \frac{H+D-2\sqrt{\left\{HD+\left(\frac{A+B}{c}\right)^2\right\}}}{H+D+2\sqrt{\left\{HD+\left(\frac{A+B}{c}\right)^2\right\}}}.$$

A discussion of the different cases that arise from giving different values to D and H is given in Legendre's "Traité des Functions Elliptiques," tome i.

Theorems in the Calculus of Operations. By J. J. WALKER.

[*Read April 8th*, 1880.]

I. The subjects of operation are functions of a single variable x, or if other variables enter these are supposed independent of x, so that x alone is considered to vary. Let u, ϕ be any such functions. The kind of operation considered is that of multiplying u by some integer power of ϕ, and then taking the differential coefficient of the product of an order differing from the index of ϕ by a given number, which may be 0, 1, 2 ... Thus the symbols for a completed set of such operations may be $Du\phi$, $D^2u\phi^2 \ldots D^ru\phi^r$, where D stands for $\frac{d}{dx}$; or $Du\phi^2$, $D^2u\phi^3 \ldots D^{r-1}u\phi^r \ldots$. A series of such terms, for shortness, may be called a *progressive series*, in the sense that the subjects of successive differentiations are not, as in Taylor's Series or Leibnitz's Theorem, one and the same function, but form a geometric progression, the common ratio being the function ϕ.

The first theorem establishes the development of $D^nu\phi^{n+1}$ in a progressive series, the terms of which are of the form $D^ru\phi^r$; viz., writing ϕ' for $D\phi$, it is proved that

$$D^nu\phi^{n+1} = \phi D^nu\phi^n + n\phi'\phi D^{n-1}u\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}\,D\phi'\phi^2\,.\,D^{n-2}u\phi^{n-2}$$
$$+ \ldots + nD^{n-2}\phi'\phi^{n-1}\,.\,Du\phi + D^{n-1}\phi'\phi^n\,.\,u \quad \ldots\ldots\ldots\ldots\ldots (a).$$

The second theorem similarly establishes the development of $D^nu\phi^{n-1}$ in another progressive series, in which the terms of the same type, $D^ru\phi^r$, are multiplied by different functional coefficients; viz., now, for convenience, writing $\psi = \phi^{-1}$, $\psi' = D\psi$,

$$D^n u\phi^{n-1} = \psi D^n u\phi^n + n\psi'\phi D^{n-1} u\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2} D\psi'\phi^2\,.\,D^{n-2}u\phi^{n-2} + \ldots$$
$$+ nD^{n-2}\psi'\phi^{n-1}\,.\,Du\phi + D^{n-1}\psi'\phi^n\,.\,u \ldots\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

This series is not a mere transformation of the former, at least I have not discovered any means of effecting such a transformation.

The third and fourth theorems are easy corollaries from the first and second; viz.,

$$\phi D\psi D^{n-1}\phi'\phi^n = \phi D^n\phi'\phi^{n-1} + n\phi'\phi D^{n-1}\phi'\phi^{n-2}$$
$$+ \frac{n\,.\,n-1}{1\,.\,2} D\phi'\phi^2\,.\,D^{n-2}\phi'\phi^{n-3} + \ldots + nD^{n-2}\phi'\phi^{n-1}\,.\,D\phi' \ \ldots\ldots\ldots(c),$$

$$\psi D\phi D^{n-1}\psi'\phi^n = \psi D^n\psi'\phi^{n+1} + n\psi'\phi D^{n-1}\psi'\phi^n$$
$$+ \frac{n\,.\,n-1}{1\,.\,2} D\psi'\phi^2\,.\,D^{n-2}\psi'\phi^{n-1} + \ldots + nD^{n-2}\psi'\phi^{n-1}\,.\,D\psi'\phi^2 \ldots\ldots(d),$$

in the latter of which the third term and the last are similar; also the fourth and the last but one, and so on. Hence it may be otherwise written

$$\psi D\phi D^{n-1}\psi'\phi^n = \psi D^n\psi'\phi^{n+1} + n\psi'\phi D^{n-1}\psi'\phi^n + \frac{n+1\,.\,n}{1\,.\,2} D\psi'\phi^2\,.\,D^{n-2}\psi'\phi^{n-1}$$
$$+ \frac{n+1\,.\,n\,.\,n-1}{1\,.\,2\,.\,3} D^2\psi'\phi^3\,.\,D^{n-3}\psi'\phi^{n-2} + \ldots \quad \ldots\ldots\ldots\ldots(d'),$$

with the obvious rule of ending with, and taking only half the coefficient of, the term $(D^{\frac{n-1}{2}}\psi'\phi^{\frac{n+1}{2}})^2$ if n is an odd integer; and, in the case of n being even, ending with that term, the succeeding one to which would be of the same form, viz., with the term $D^{\frac{n-2}{2}}\psi'\phi^{\frac{n}{2}}\,.\,D^{\frac{n}{2}}\psi'\phi^{\frac{n+2}{2}}$.

In connexion with my original proof of the equality (b), I found that

$$\psi D\psi D^{n-1}\phi'\phi^n + n\psi'\phi D\psi D^{n-2}\phi'\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2} D\psi'\phi^2\,.\,D\psi D^{n-3}\phi'\phi^{n-2} + \ldots$$
$$\frac{n\,.\,n-1}{1\,.\,2} D^{n-3}\psi'\phi^{n-2}\,.\,D\psi D\phi'\phi^2 + nD^{n-2}\psi'\phi^{n-1}\,.\,D\phi' = \psi D^n\phi'\phi^{n-1} \quad \ldots(e),$$

$$\phi D\phi D^{n-1}\psi'\phi^n + n\phi'\phi D\phi D^{n-2}\psi'\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2} D\phi'\phi^2\,.\,D\phi D^{n-3}\psi'\phi^{n-2} + \ldots$$
$$+ \frac{n\,.\,n-1}{1\,.\,2} D^{n-3}\phi'\phi^{n-2}\,.\,D\phi D\psi'\phi^2 + nD^{n-2}\phi'\phi^{n-1}\,.\,D\psi'\phi^2 = \phi D^n\psi'\phi^{n+1} \ \ldots(f),$$

—relations, of which, although I have not found them otherwise useful, I give demonstrations.

II. I am indebted to Mr. J. Hammond for a remark—not less valuable because apparently very obvious when made—which is, in substance, that, by changing ϕ into v and ϕ' into v' in certain places, the proof which I originally gave to establish the first of the above theorems, establishes the more general one, v being any third function of x,

$$D^n uv\phi^n = vD^n u\phi^n + nv'\phi D^{n-1}u\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}\,Dv'\phi^2\,.\,D^{n-2}u\phi^{n-2} + \dots$$
$$+\frac{n\,.\,n-1}{1\,.\,2}\,D^{n-3}v'\phi^{n-2}\,.\,D^2u\phi^2 + nD^{n-2}v'\phi^{n-1}\,.\,Du\phi + D^{n-1}v'\phi^n\,.\,u \quad \dots(g),$$

which, by making $v = \phi$, gives the expansion (a), and by making $v = \phi^{-1} = \psi$, gives (b); thus enabling me to dispense with a rather tedious independent proof of (b). The original proof of (a) thus modified, is as follows. Writing also

$$v' = Dv \text{ as } \phi' = D\phi,$$

it is readily verified that

$$Duv\phi = vDu\phi + v'\phi\,.\,u \dots\dots\dots\dots\dots\dots(1);$$

or, writing $u\phi$ for u, $Duv\phi^2 = vDu\phi^2 + v'\phi\,.\,u\phi$.

Differentiating this equation,

$$\begin{aligned} D^2uv\phi^2 &= vD^2u\phi^2 + v'\phi Du\phi \\ &\qquad\qquad + v'\ Du\phi^2 + \phi Dv'\phi\,.\,u \\ &= vD^2u\phi^2 + v'\phi Du\phi \\ &\qquad\qquad + v'\phi Du\phi + (\phi Dv'\phi + \phi'\phi v')\,u \\ &= vD^2u\phi^2 + 2v'\phi Du\phi + Dv'\phi^2\,.\,u \dots\dots\dots\dots(2), \end{aligned}$$

in virtue of the equalities

$$Du\phi^2 = \phi Du\phi + \phi'\phi\,.\,u,\quad \phi Dv'\phi + \phi'\phi v' = Dv'\phi^2,$$

which follow from (1) by giving v the particular value ϕ; and u, v the particular values v', ϕ successively.

Now assume that, for all positive integers up to and including r,

$$D^r uv\phi^r = vD^r u\phi^r + rv'\phi D^{r-1}u\phi^{r-1} + \frac{r\,.\,r-1}{1\,.\,2}\,Dv'\phi^2 D^{r-2}u\phi^{r-2} + \dots$$
$$+ rD^{r-2}v'\phi^{r-1}Du\phi + D^{r-1}v'\phi^r\,.\,u \dots\dots\dots\dots(3);$$

then, substituting $u\phi$ for u, it will follow that

$$D^r uv\phi^{r+1} = vD^r u\phi^{r+1} + rv'\phi D^{r-1}u\phi^r + \frac{r\,.\,r-1}{1\,.\,2}\,Dv'\phi^2\,.\,D^{r-2}u\phi^{r-1} + \dots$$
$$+ rD^{r-2}v'\phi^{r-1}\,.\,Du\phi^2 + D^{r-1}v'\phi^r\,.\,u\phi.$$

Differentiating this equality, and observing that

$$D^r u\phi^{r+1} = \phi D^r u\phi^r + r\phi'\phi D^{r-1}u\phi^{r-1} + \dots$$

by making $v = \phi$ in (3),

$$\begin{aligned} D^{r+1}uv\phi^{r+1} = vD^{r+1}u\phi^{r+1} + rv'\phi D^r u\phi^r + \frac{r\,.\,r-1}{1\,.\,2}\,Dv'\phi^2\,.\,D^{r-1}u\phi^{r-1} + \dots \\ + D^{r-1}v'\phi^r\,.\,Du\phi \\ + v'\left(\phi D^r u\phi^r + \frac{r}{1}\,\phi'\phi D^{r-1}u\phi^{r-1} + \frac{r\,.\,r-1}{1\,.\,2}\,D\phi'\phi^2\,.\,D^{r-2}u\phi^{r-2} + \dots \right. \\ \left. + D^{r-1}\phi'\phi^r\,.\,u\right) \\ + rDv'\phi\left(\phi D^{r-1}u\phi^{r-1} + \frac{r-1}{1}\,\phi'\phi D^{r-2}u\phi^{r-2} \right. \\ \left. + \frac{r-1\,.\,r-2}{1\,.\,2}\,D\phi'\phi^2\,.\,D^{r-3}u\phi^{r-3} + \dots\right) \\ + \frac{r\,.\,r-1}{1\,.\,2}\,D^2v'\phi^2\left(\phi D^{r-2}u\phi^{r-2} + \frac{r-2}{1}\,\phi'\phi D^{r-3}u\phi^{r-3} + \dots\right) \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ + rD^{r-1}v'\phi^{r-1}\left(\phi Du\phi + \phi'\phi u\right) + \phi D^r v'\phi^r\,.\,u. \end{aligned}$$

In this development the coefficient of $D^r u\phi^r$ is $(r+1)\,v'\phi$; that of $D^{r-1}u\phi^{r-1}$ is $\frac{r\,.\,r-1}{1\,.\,2}\,Dv'\phi^2 + r\,(\phi Dv'\phi + \phi'\phi\,.\,v')$, or $\frac{r+1\,.\,r}{1\,.\,2}\,Dv'\phi^2$ by (1); that of $D^{r-2}u\phi^{r-2}$ is

$$\frac{r\,.\,r-1\,.\,r-2}{1\,.\,2\,.\,3}\,D^2v'\phi^3 + \frac{r\,.\,r-1}{1\,.\,2}\,(\phi D^2v'\phi^2 + 2\phi\phi' Dv'\phi + D\phi'\phi^2\,.\,v'),$$

or

$$\frac{r+1\,.\,r\,.\,r-1}{1\,.\,2\,.\,3}\,D^2v'\phi^3 \text{ by } (2);$$

and, generally, the coefficient of $D^{r-p}u\phi^{r-p}$, p being any positive integer not greater than r, is

$$\frac{r\,.\,r-1\,\dots\,r-p}{1\,.\,2\,\dots\,p+1}\,D^p\phi'\phi^{p+1} + \frac{r\,.\,r-1\,\dots\,r-p+1}{1\,.\,2\,\dots\,p}$$

$$\begin{aligned} \left(\phi D^p v'\phi^p + p\phi'\phi D^{p-1}v'\phi^{p-1} + \frac{p\,.\,p-1}{1\,.\,2}\,D\phi'\phi^2\,.\,D^{p-2}v'\phi^{p-2}\dots\right. \\ \left.\dots + pD^{p-2}\phi'\phi^{p-1}Dv'\phi + D^{p-1}\phi'\phi^p\,.\,v'\right), \end{aligned}$$

wherein the terms within brackets are equal to $D^p v'\phi^{p+1}$, by the assumption that the formula (3) holds for all integers up to and including r, whatever u or v is, and therefore when, in particular, $u = v'$, $v = \phi$. Thus the coefficient of $D^{r-p}u\phi^{r-p}$ is equal to

$$\left(\frac{r\,.\,r-1\,\dots\,r-p}{1\,.\,2\,\dots\,p+1} + \frac{r\,.\,r-1\,\dots\,r-p+1}{1\,.\,2\,\dots\,p}\right) D^p v'\phi^{p+1},$$

or to

$$\frac{r+1\,.\,r\,.\,r-1\,\dots\,r-p+1}{1\,.\,2\,.\,3\,\dots\,p\,.\,p+1}\,D^p v'\phi^{p+1}.$$

It appears therefore that, if the law of development expressed in (3) holds for all integers up to and including r, it will hold for $r+1$. But it has been verified for $r = 1$, $r = 2$, and therefore holds generally.

III. Theorems (*a*) and (*b*) having been thus established, consider the equality

$$D\phi^{-1}D^{n-1}\phi'\phi^n = \phi^{-1}D^n\phi'\phi^n - D^{n-1}\phi'\phi^n \,.\, \phi'\phi^{-2},$$

or

$$\phi D\phi^{-1}D^{n-1}\phi'\phi^n = D^n\phi'\phi^n - D^{n-1}\phi'\phi^n \,.\, \phi'\phi^{-1};$$

which gives, by writing $\phi'\phi^{-1}$ for u in (*a*),

$$\phi D\phi^{-1}D^{n-1}\phi'\phi^n = \phi D^n\phi'\phi^{n-1} + n\phi'\phi D^{n-1}\phi'\phi^{n-2}$$
$$+ \frac{n\,.\,n-1}{1\,.\,2} D\phi'\phi^2 \,.\, D^{n-1}\phi'\phi^{n-3} + \ldots + nD^{n-1}\phi'\phi^{n-1} \,.\, D\phi',$$

which is theorem (*c*).

Similarly

$$D\phi D^{n-1}\psi'\phi^n = \phi D^n\psi'\phi^n - D^{n-1}\psi'\phi^n \,.\, \psi'\phi^2,$$

or

$$\psi D\phi D^{n-1}\psi'\phi^n = D^n\psi'\phi^n - D^{n-1}\psi'\phi^{n-1} \,.\, \psi'\phi;$$

which gives, by writing $\psi'\phi$ for u in (*b*),

$$\psi D\phi D^{n-1}\psi'\phi^n = \psi D^n\psi'\phi^{n+1} + n\psi'\phi D^{n-1}\psi'\phi^n + \frac{n\,.\,n-1}{1\,.\,2} D\psi'\phi^2 D^{n-2}\psi'\phi^{n-1} + \ldots$$
$$+ nD^{n-2}\psi'\phi^{n-1} \,.\, D\psi'\phi^2,$$

which is theorem (*d*).

The remark is now obvious, that both expansions (*c*), (*d*) are contained in the more general one

$$vDv^{-1}D^{n-1}v'\phi^n = D^nv'\phi^n - D^{n-1}v'\phi^n \,.\, v'v^{-1}$$
$$= vD^nv'v^{-1}\phi^n + nv'\phi D^{n-1}v'v^{-1}\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2} Dv'\phi^2 D^{n-2}v'v^{-1}\phi^{n-2} + \ldots$$
$$+ nD^{n-2}v'\phi^{n-1}Dv'v^{-1}\phi,$$

which follows from (*g*), by making $u = v'v^{-1}$; but the expansion (*d*) or (*d'*) is that which I have found of most fruitful application.

IV. Finally, to prove the relation (*e*), it follows from (*c*) that

$$\begin{aligned}
D\psi D^{n-1}\phi'\phi^n &= D^n\phi'\phi^{n-1} + n\psi\phi'\phi D^{n-1}\phi'\phi^{n-2} \\
&\quad + \frac{n\,.\,n-1}{1\,.\,2}\psi D\phi'\phi^2 \,.\, D^{n-2}\phi'\phi^{n-3} + \ldots + n\psi D^{n-2}\phi'\phi^{n-1} \,.\, D\phi', \\
D\psi D^{n-2}\phi'\phi^{n-1} &= \ldots\ldots\ldots\ldots\ldots\ldots D^{n-1}\phi'\phi^{n-2} \\
&\quad + (n-1)\,\psi\phi'\phi D^{n-2}\phi'\phi^{n-3} + \ldots + (n-1)\psi D^{n-3}\phi'\phi^{n-2} \,.\, D\phi', \\
D\psi D^{n-3}\phi'\phi^{n-2} &= \ldots\ldots\ldots \\
&\quad \ldots\ldots\ldots D^{n-2}\phi'\phi^{n-3} + \ldots + (n-2)\psi D^{n-4}\phi'\phi^{n-3} \,.\, D\phi', \\
\ldots \quad \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
D\psi D\phi'\phi^2 &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad D^2\phi'\phi + 2\psi\phi'\phi D\phi', \\
D\psi\phi'\phi &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad D\phi';
\end{aligned}$$

and hence

$$\psi D\psi D^{n-1}\phi'\phi^n + n\psi'\phi D\psi D^{n-2}\phi'\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2} D\psi'\phi^2 \,.\, D\psi D^{n-3}\phi'\phi^{n-2} + \ldots$$
$$+ \frac{n\,.\,n-1}{1\,.\,2} D^{n-3}\psi'\phi^{n-2} \,.\, D\psi D\phi'\phi^2 + nD^{n-2}\psi'\phi^{n-1}D\phi' = \psi D^n\phi'\phi^{n-1}$$

$$+n(\phi'+\psi'\phi^2)\psi D^{n-1}\phi'\phi^{n-2}+\frac{n.n-1}{1.2}(\psi D\phi'\phi^2+2\psi'\phi\phi'\phi+\phi D\psi'\phi^2)\psi D^{n-2}\phi'\phi^{n-3}$$

$$+\ldots+n\left(\psi D^{n-2}\phi'\phi^{n-1}+(n-1)\psi'\phi D^{n-3}\phi'\phi^{n-2}\right.$$
$$+\frac{n-1.n-2}{1.2}D\psi'\phi^2 D^{n-4}\phi'\phi^{n-3}+\ldots$$
$$\left.+(n-1)\,D^{n-3}\psi'\phi^{n-2}.\phi'\phi+\phi D^{n-2}\psi'\phi^{n-1}\right)\psi D\phi',$$

wherein every group of terms inclosed by brackets vanishes separately.

For, taking that forming the coefficient of $\psi D\phi'$ as the general form, n being any positive integer, in virtue of the well-known property of binomial coefficients, viz., $\frac{(n-1)(n-2)\ldots(n-p)}{1.2\ldots p}$

$$=\frac{(n-2)(n-3)\ldots(n-p-1)}{1.2\ldots p}+\frac{(n-2)(n-3)\ldots(n-p)}{1.2\ldots(p-1)},$$

this coefficient is equal to

$$\psi D^{n-2}\phi'\phi^{n-1}+(n-2)\psi'\phi D^{n-3}\phi'\phi^{n-2}+\frac{(n-2)(n-3)}{1.2}D\psi'\phi^2.D^{n-4}\phi'\phi^{n-3}+\ldots$$
$$+D^{n-3}\psi'\phi^{n-2}.\phi'\phi$$
$$+D^{n-3}\phi'\phi^{n-2}.\psi'\phi+(n-2)\,D^{n-4}\phi'\phi^{n-3}.D\psi'\phi^2+\ldots$$
$$+(n-2)\,\phi'\phi D^{n-3}\psi'\phi^{n-2}+\phi D^{n-2}\psi'\phi^{n-1};$$

or, in virtue of (b), (a), by changing n into $n-2$ in both, and making $u=\phi'\phi$, $\psi'\phi$ in the former and latter respectively, it is equal to

$$D^{n-2}\phi'\phi^{n-2}+D^{n-2}\psi'\phi^n \text{ or } D^{n-2}\phi^{n-2}(\phi'+\psi'\phi^2),$$

which vanishes, since $\psi=\phi^{-1}$.

Thus formula (e) is established, and, in a precisely analogous way, formula (f) may also be demonstrated.

Hereafter I hope to lay before the Society some applications of the foregoing theorems.

The Calculus of Equivalent Statements. (*Fourth Paper.*)

By HUGH McCOLL, B.A.

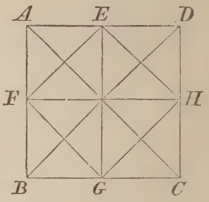
[*Read Feb.* 12*th*, 1880, *revised April* 22*nd*, 1880.]

Probability Notation.

In applying symbolical logic to probability, the following notation will, I think, be found useful.

DEF. 1.—The symbol x_a denotes the *chance* that the statement x is true *on the assumption that the statement a is true.*

As illustrations we may take $x_x = 1$, $x_{x'} = 0$, $x'_x = 0$, $(a+x)_{ax} = 1$, $(ax)_{a'+x'} = 0$.



Again, speaking of a point taken at random within the square *ABCD*, the sides of which are bisected at the points *E*, *F*, *G*, *H*, let *a* denote the statement, "It will fall within the triangle *ABD*," and let *x* denote the statement, "It will fall within the square *EFGH*." Then, evidently, $x_a = \frac{1}{2}$. Again, speaking of the same random point, let *y* denote the statement, "It will fall within the triangle *EHD*." Then $y_a = \frac{1}{8}$, $y_x = 0$, $y_{x'} = \frac{1}{4}$, $x_y = 0$, $x_{y'} = \frac{4}{7}$, $y'_x = 1$, $(ax)_{y'} = \frac{2}{7}$.

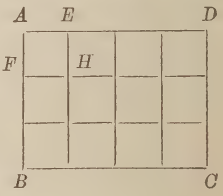
Def. 2.—The symbol ϵ is an equivalent for the symbol 1, and denotes any statement whose truth is taken for granted *throughout the whole of an investigation.*

Hence x_ϵ simply denotes *the chance that x is true.*

In the geometrical illustration to Def. 1, ϵ would denote the statement, "The random point will fall within the square *ABCD*;" and, attaching the same meanings as before to the other letters, we have $x_\epsilon = \frac{1}{2}$, $(ax)_\epsilon = \frac{1}{4}$, $a_\epsilon = \frac{1}{2}$, $(a+x)_\epsilon = \frac{3}{4}$, $(a'x)_\epsilon = \frac{1}{4}$, $(ax')_\epsilon = \frac{1}{4}$.

Note.—It is worth noticing that the implication $a : x$ is equivalent to the statement $x_a = 1$.

Def. 3.—Any two statements *a* and *x* are said to be *independent*, when we have any one of the equivalent equations $x_a = x_\epsilon$, $a_x = a_\epsilon$, $x_a = x_{a'}$, $a_x = a_{x'}$. (See Appendix, Note 1.)



Independent statements may be illustrated geometrically thus. Suppose a point to be taken at random within the rectangle *ABCD*, which is made up of 12 equal squares, 4 in each row, and 3 in each column. Let *a* denote the statement, "The random point will fall within the rectangle *DF*;" and let *x* denote the statement, "The random point will fall within the rectangle *BE*." Then, evidently, *a* and *x* are independent, for

$$a_x = AH \div BE = \tfrac{1}{3},\quad a_\epsilon = DF \div AC = \tfrac{1}{3},\quad a_{x'} = DH \div EC = \tfrac{1}{3};$$

$$x_a = AH \div DF = \tfrac{1}{4},\quad x_\epsilon = BE \div AC = \tfrac{1}{4},\quad x_{a'} = BH \div FC = \tfrac{1}{4}.$$

Let $AB = m$, and $BC = n$; and let *x*, *y* be the coordinates of a random point in the rectangle *AC*, referred to *AB* and *BC* as axes. Then, supposing the rectangle *AC* to be divided into $\frac{mn}{dx\,dy}$ equal parts, each part being an infinitesimal rectangle whose area is $dx\,dy$, the chance that the random point will fall within some particular one of these rectangles, is $\frac{dx\,dy}{mn}$; the chance that the infinitesimal rectangle within which it falls is at a distance x from AB is $\frac{dx}{n}$, and the chance that it is at a distance y from BC is $\frac{dy}{m}$. It is clear that the statement, the chance of whose truth is $\frac{dx}{n}$, in no way affects the chance of the truth of the statement whose chance is $\frac{dy}{m}$, so that the chance of the truth of these two independent statements combined is $\frac{dx\,dy}{mn}$. If these two independent statements be denoted by x and y respectively, it may be shown, by the same reasoning as before, that $x_y = x_\epsilon = x_{y'}$, and that $y_x = y_\epsilon = y_{x'}$.

Rule 1.—$(ab)_x = a_x b_{ax}$, $(abc)_x = a_x b_{ax} c_{abx}$, and so on, for any compound statement. Putting ϵ for x in these equations, we get

$$(ab)_\epsilon = a_\epsilon b_a, \quad (abc)_\epsilon = a_\epsilon b_a c_{ab},$$

and so on. (See Note after Rule 2.)

Note.—From this rule, we see that $x_a = \frac{(ax)_\epsilon}{a_\epsilon}$, and therefore $x_a \div a_x = x_\epsilon \div a_\epsilon$.

Rule 2.— $$(a+b)_x = a_x + b_x - (ab)_x;$$

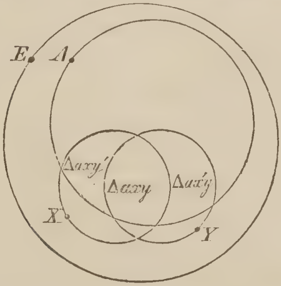
$$(a+b+c)_x = (a_x+b_x+c_x) - \{(ab)_x + (ac)_x + (bc)_x\} + (abc)_x;$$

and, generally,

$$(a_1+a_2+a_3+\ldots+a_n)_x = s_1 - s_2 + s_3 - \ldots + (-1)^{n-1} s_n,$$

in which s_r denotes the sum of the $\frac{|\underline{n}}{|\underline{r}\;|\underline{n-r}}$ chances (the statement x being taken for granted) of the $\frac{|\underline{n}}{|\underline{r}\;|\underline{n-r}}$ combinations that can be made of the n statements $a_1, a_2, a_3 \ldots a_n$, taken r at a time.

Note.—The following geometrical illustrations will help to illustrate and verify Rules 1 and 2. Let ϵ, a, x, y respectively denote the four



statements—(1) *A point is taken at random within the circle E*, (2) *It* *falls within the circle* A, (3) *It falls within the circle* X, (4) *It falls within the circle* Y. Let $\Delta\epsilon$, Δx, Δax, Δaxy, $\Delta ax'y$, $\Delta axy'$, $\Delta(x+y)$ respectively denote the *areas* with reference to which the *statements* ϵ, x, ax, axy, $ax'y$, axy', $x+y$ respectively, can possibly be true; and, similarly, let the symbol $\Delta f(\epsilon, a, x, y)$ generally denote the area with reference to which the statement $f(\epsilon, a, x, y)$ can possibly be true. Then

$$(xy)_a = \frac{\Delta axy}{\Delta a} = \frac{\Delta ax}{\Delta a} \times \frac{\Delta axy}{\Delta ax} = x_a y_{ax},$$

$$\begin{aligned}(x+y)_a = \frac{\Delta(ax+ay)}{\Delta a} &= \frac{\Delta(axy'+axy+ax'y)}{\Delta a}\\ &= \frac{\Delta axy' + \Delta axy + \Delta ax'y}{\Delta a}\\ &= \frac{\Delta ax + \Delta ay - \Delta axy}{\Delta a}\\ &= \frac{\Delta ax}{\Delta a} + \frac{\Delta ay}{\Delta a} - \frac{\Delta axy}{\Delta a}\\ &= x_a + y_a - (xy)_a.\end{aligned}$$

The above expressions are of course simplified, when for a we put ϵ. In this case, we have

$$(xy)_\epsilon = \frac{\Delta xy}{\Delta\epsilon} = \frac{\Delta x}{\Delta\epsilon} \times \frac{\Delta xy}{\Delta x} = x_\epsilon y_x,$$

$$\begin{aligned}(x+y)_\epsilon = \frac{\Delta(x+y)}{\Delta\epsilon} &= \frac{\Delta(xy'+xy+x'y)}{\Delta\epsilon}\\ &= \frac{\Delta xy' + \Delta xy + \Delta x'y}{\Delta\epsilon}\\ &= \frac{\Delta x + \Delta y - \Delta xy}{\Delta\epsilon}\\ &= \frac{\Delta x}{\Delta\epsilon} + \frac{\Delta y}{\Delta\epsilon} - \frac{\Delta xy}{\Delta\epsilon}\\ &= x_\epsilon + y_\epsilon - (xy)_\epsilon.\end{aligned}$$

RULE 3.— $$(x : a) : \left(x_a = \frac{x_\epsilon}{a_\epsilon}\right).$$

This follows readily from Rule 1. For, since $(ax)_\epsilon = a_\epsilon x_a$, we have $x_a = \frac{(ax)_\epsilon}{a_\epsilon}$. But $x = ax$ by supposition, so that $x_a = \frac{x_\epsilon}{a_\epsilon}$.

NOTE.—From this we see that $x_{x+y} = \frac{x_\epsilon}{(x+y)_\epsilon}$, and that $(xy)_x = y_x$, whatever be the statements x and y.

RULE 4.—Taking $(a-ab)_{\cdot x}$ or $\{a(1-b)\}_{\cdot x}$ as a convenient abbreviation for $a_x-(ab)_x$, and so on for similar expressions, we may clear of accents a compound chance of the form $(abc \ldots d'e'f' \ldots)_x$ by means of the equation

$$(abc \ldots d'e'f' \ldots)_x = \{(abc \ldots)(1-d)(1-e)(1-f) \ldots\}_{\cdot x},$$

the factors on the right-hand side of the equation to be multiplied like ordinary algebraical factors, and then each term of the product to be affected by the suffix x.

Beginning with the simplest case, we have

$$\begin{aligned}(\alpha\beta')_x &= \alpha_x \beta'_{\alpha x} = \alpha_x (1-\beta_{\alpha x}) \\ &= \alpha_x - \alpha_x \beta_{\alpha x} = \alpha_x - (\alpha\beta)_x \\ &= (\alpha - \alpha\beta)_{\cdot x} = \{\alpha (1-\beta)\}_{\cdot x}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\alpha\beta'\gamma')_x &= \alpha_x (\beta'\gamma')_{\alpha x} = \alpha_x \{1-(\beta+\gamma)_{\alpha x}\} \\ &= \alpha_x \{1-\beta_{\alpha x} - \gamma_{\alpha x} + (\beta\gamma)_{\alpha x}\} \\ &= \alpha_x - \alpha_x \beta_{\alpha x} - \alpha_x \gamma_{\alpha x} + \alpha_x (\beta\gamma)_{\alpha x} \\ &= \alpha_x - (\alpha\beta)_x - (\alpha\gamma)_x + (\alpha\beta\gamma)_x \\ &= (\alpha - \alpha\beta - \alpha\gamma + \alpha\beta\gamma)_{\cdot x} \\ &= \{\alpha (1-\beta)(1-\gamma)\}_{\cdot x}.\end{aligned}$$

And the same method of proof may evidently be extended to a compound statement with any number of accented factors.

To illustrate the use of the preceding rules and notation, I will apply them to two problems taken from Boole's "Laws of Thought." In the first of these my result agrees with that of Boole, but in the second our results differ.

The chance that a witness A. speaks the truth is p, the chance that another witness B. speaks the truth is q, and the chance that they disagree in a statement is r. What is the chance that, if they agree, their statement is true? (See Boole's "Laws of Thought," p. 279.)

Let x denote the statement, "A. speaks the truth," and let y denote the statement, "B. speaks the truth."

The data are $x_\epsilon = p$, $y_\epsilon = q$, $(x'y + xy')_\epsilon = r$; and we are required to find $(xy)_{xy+x'y'}$.

Now $$(xy)_{xy+x'y'} = \frac{(xy)_\epsilon}{(xy+x'y')_\epsilon} \text{ (See Rule 3, Note)}$$
$$= \frac{(xy)_\epsilon}{1-r};$$

and, to find $(xy)_\epsilon$, we have

$$\begin{aligned}r = (x'y + xy')_\epsilon &= (x'y)_\epsilon + (xy')_\epsilon \\ &= \{(1-x)\,y\}_{\cdot\epsilon} + \{x\,(1-y)\}_{\cdot\epsilon} \text{ (from Rule 4)} \\ &= x_\epsilon + y_\epsilon - 2\,(xy)_\epsilon = p + q - 2\,(xy)_\epsilon.\end{aligned}$$

Therefore $$(xy)_\epsilon = \tfrac{1}{2}(p+q-r),$$

and, substituting, we get $(xy)_{xy+x'y'} = \dfrac{p+q-r}{2(1-r)}$.

NOTE.—It is worth noticing that in this solution no assumption is made as to whether or not the statements x and y are independent.

The next problem is this :—

The chances of two causes A_1 and A_2 are c_1 and c_2 respectively. The chance that, if the cause A_1 present itself, an event E will accompany it, whether as a consequence of the cause A_1 or not, is p_1; and the chance that, if the cause A_2 present itself, the event E will accompany it, whether as a consequence of it or not, is p_2. Moreover, the event E cannot appear in the absence of both the causes A_1 and A_2. Required the chance of the event E. (See Boole's "Laws of Thought," p. 321.)

Let $x =$ the cause A_1 will present itself,
let $y =$ the cause A_2 will present itself,
and let $z =$ the event E will occur.

The data are $x_\epsilon = c_1$, $y_\epsilon = c_2$, $z_x = p_1$, $z_y = p_2$, and the implication $z : x+y$; and we are required to find z_ϵ.

Now the given implication $z : x+y$ is equivalent to the equation $z = z(x+y)$. Hence

$$\begin{aligned} z_\epsilon &= (xz+yz)_\epsilon = (xz)_\epsilon + (yz)_\epsilon - (xyz)_\epsilon \text{ (by Rule 2)} \\ &= x_\epsilon z_x + y_\epsilon z_y - (xy)_\epsilon z_{xy} \text{ (by Rule 1)} \\ &= c_1 p_1 + c_2 p_2 - (xy)_\epsilon z_{xy}. \end{aligned}$$

If (guided by the concrete illustration in Boole's foot-note, see Appendix, Note 2) we assume A_1 and A_2 to be independent, and the event E to be more probable when both causes exist, than when only one of them exists, we shall have $(xy)_\epsilon = x_\epsilon y_\epsilon = c_1 c_2$, and z_{xy} greater than z_x and greater than z_y, that is, greater than p_1 and greater than p_2. Hence $c_1p_1 + c_2p_2 - c_1c_2p_1$ and $c_1p_1 + c_2p_2 - c_1c_2p_2$ are superior limits to z_ϵ, while (since 1 is a superior limit to z_{xy}) $c_1p_1 + c_2p_2 - c_1c_2$ is evidently an inferior limit to z_ϵ.

If we take $c_1 = {\cdot}1$, $c_2 = {\cdot}2$, $p_1 = {\cdot}6$, and $p_2 = {\cdot}7$, we shall find z_ϵ to be within the limits ·18 and ·186.

Boole's equation* for finding the *exact* value of the required chance gives (on substituting these numerical values of c_1, c_2, p_1, p_2) ·19069, &c., as the required result, which is too much by at least ·00469...

RULE 5.—From the compound implication $(\alpha : x)(x : \beta)$ we may infer that α_ϵ is an inferior limit of x_ϵ, and β_ϵ a superior limit.

This equation (in which u denotes the required chance) is

$$\frac{(u-c_1p_1)(u-c_2p_2)}{c_1p_1+c_2p_2-u} = \frac{\{1-c_1(1-p_1)-u\}\{1-c_2(1-p_2)-u\}}{1-u}.$$

This rule is useful in such cases as the following: Suppose we have, among other data, a series of implications, each of the form $f(x) : \phi(x)$, and that we are required to find the limits of x_ϵ. Solving with respect to x, the given implications combined will be equivalent to the compound implication $(\alpha : x)(x : \beta)(\gamma : 0)$, in which α denotes the weakest antecedent of x cleared of zero terms, β the strongest consequent of x cleared of zero terms, and γ the sum of the zero terms which do not involve x. The solution can hardly be called complete till α, β, γ (if complex disjunctive statements) are reduced to their primitive forms. (See Rule 21 of my Third Paper.)

Question 6258, proposed in the "Educational Times" for last March, may be taken as a simple illustration of the use of this rule. In this question the given compound implication

$$(ax+bx' : c)(bx+ax' : c')$$

is equivalent to $(ac+bc' : x)(x : a'c'+b'c)(ab : 0)$.

Appendix.

NOTE 1.—To prove that (as stated in Def. 3) the four equations $x_a = x_\epsilon$, $a_x = a_\epsilon$, $x_a = x_{a'}$, and $a_x = a_{x'}$ are equivalent statements, each implying the rest.

From Rule 1, we have $x_\epsilon a_x = a_\epsilon x_a$; consequently, if we assume $x_a = x_\epsilon$, we shall have $a_x = a_\epsilon$; and, conversely, if we assume $a_x = a_\epsilon$, we shall have $x_a = x_\epsilon$.

We will next assume the equation $a_x = a_{x'}$, and from it deduce the equation $x_a = x_\epsilon$.

From Rule 1, we get $a_x = \dfrac{a_\epsilon x_a}{x_\epsilon}$, and $a_{x'} = \dfrac{a_\epsilon x'_a}{x'_\epsilon}$. But, by supposition, $a_x = a_{x'}$; therefore

$$\frac{x_a}{x_\epsilon} = \frac{x'_a}{x'_\epsilon};$$

that is,

$$\frac{x_a}{x_\epsilon} = \frac{1-x_a}{1-x_\epsilon};$$

whence we get

$$x_a = x_\epsilon.$$

By reversing this proof, we may from the equation $x_a = x^\epsilon$ deduce the equation $a_x = a_{x'}$.

We have thus proved the three equations $a_x = a_\epsilon$, $x_a = x_\epsilon$, $a_x = a_{x'}$ to be equivalent.

In the same way as it was shown that $a_x = a_{x'}$ was equivalent to $x_a = x_\epsilon$, we may show that $x_a = x_{a'}$ is equivalent to $a_x = a_\epsilon$. Hence we finally get $(x_a = x_\epsilon) = (a_x = a_\epsilon) = (x_a = x_{a'}) = (a_x = a_{x'})$.

NOTE 2.—The foot-note referred to, page 118, is the following:—

The mode in which such data as the above might be furnished by experience, is easily conceivable. Opposite the window of the room in which I write, is a field, liable to be over-flowed from two causes, distinct, but capable of being combined, viz., floods from the upper sources of the River Lee, and tides from the ocean. Suppose that observations made on N separate occasions have yielded the following results: on A occasions the river was swollen by freshets, and on P of those occasions it was inundated, whether from this cause or not. On B occasions the river was swollen by the tide, and on Q of those occasions it was inundated, whether from this cause or not. Supposing, then, that the field cannot be inundated in the absence of *both* the causes above mentioned, let it be required to determine the total probability of its inundation.

Here the elements a, b, p, q of the general problem represent the ratios $\frac{A}{N}$, $\frac{P}{A}$, $\frac{B}{N}$, $\frac{Q}{B}$, or rather the values to which those ratios approach, as the value of N is indefinitely increased. (Boole's "Laws of Thought," p. 321.)

[The following is a very simple proof of the fundamental rule in the Inverse Method of Probability.

Let x assert the occurrence of a certain event, and let $\alpha, \beta, \gamma, \ldots$ respectively assert the existence of several mutually exclusive circumstances, any one of which may be the cause of the event. We have to prove that $\alpha_x = \dfrac{x_\alpha}{x_\alpha + x_\beta + x_\gamma + \ldots}$, on the assumption that $x : \alpha + \beta + \gamma + \ldots$, and that $\alpha_\epsilon = \beta_\epsilon = \gamma_\epsilon = \&\text{c.}$

Since $$(\alpha x)_\epsilon = x_\epsilon \alpha_x = \alpha_\epsilon x_\alpha,$$

therefore $$\alpha_x = \frac{\alpha_\epsilon x_\alpha}{x_\epsilon}.$$

But $x_\epsilon = (\alpha x + \beta x + \gamma x + \ldots)_\epsilon$, since $x : \alpha + \beta + \gamma + \ldots$, that is,

$$\begin{aligned} x_\epsilon &= (\alpha x)_\epsilon + (\beta x)_\epsilon + (\gamma x)_\epsilon + \ldots \\ &= \alpha_\epsilon x_\alpha + \beta_\epsilon x_\beta + \gamma^\epsilon x_\gamma + \ldots \end{aligned}$$

Substituting this value of x_ϵ, we get

$$\alpha_x = \frac{\alpha_\epsilon x_\alpha}{\alpha_\epsilon x_\alpha + \beta_\epsilon x_\beta + \gamma_\epsilon x_\gamma + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But, by supposition, $\alpha_\epsilon = \beta_\epsilon = \gamma_\epsilon = \&\text{c.}$; therefore

$$\alpha_x = \frac{x_\alpha}{x_\alpha + x_\beta + x_\gamma + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The same rule, of course, holds good when we interchange α and β, α and γ, and so on. The expression (1) is, of course, more general than (2), and should be used when we cannot legitimately assume that $\alpha_\epsilon = \beta_\epsilon = \gamma_\epsilon = \&\text{c.}$]

In concluding this paper I wish to record my obligations to Mr. C. J. Monro, of Hadley, Barnet, for much kind and patient criticism of my method generally, as well as for some valuable suggestions as to the directions which my investigations should take. It was by his advice that I resumed the study of probability, in which my method first originated; and it was in working out a problem in probability, which he proposed to me as a test of the power of my method, that I hit upon the notation which I here propose.

May 13*th*, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The following communications were made:—

"On Cremonian Congruences:" Dr. Hirst, F.R.S.

"On some Statical and Kinematical Theorems:" Prof. Minchin.

"On a Class of Analytical Problems:" Prof. Cayley, F.R.S.

The following presents were made to the Library:—

"Educational Times," May, 1880.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xv., Livraisons 1, 2; Harlem, 1880.

"Journal de l'Ecole Polytechnique," xlvi. Cahier, Tome xxviii.; Paris, 1879.

"Monatsbericht," Jan., Feb., 1880.

"Atti della Reale Accademia dei Lincei—Transunti," Fasc. 5, Aprile 1880, Vol. iv.; Roma, 1880.

"Proceedings of Royal Society," Vol. xxx., No. 203.

"Crelle," Band 89, 2^{es} Heft; Berlin, 1880.

"Sur les Systèmes desmiques de trois tétraèdres," par M. Cyparissos Stephanos," (Extrait du Bulletin des Sciences Math. et Astr., 2^{e} Série, Tome iii., 1879, pp. 1—33): from the Author.

"Chronik des von dem Herrn Professor Schellbach geletteten Mathematisch-Pädagogischen Seminars, 1855—1880, zur Jubelfeier seines fünf-und-zwanzigjährigen Bestehens entworfen," von Felix Müller; Berlin, 1880.

"Sur l'application d'un principe de la théorie des fonctions à des recherches purement géométriques," par M. Elling Holst. (Extrait du Bulletin de la Société Mathématique de France, Tome viii., 1880, pp. 1—8.)

"On a new Method of obtaining the Differentials of Functions, with especial reference to the Newtonian Conception of Rates or Velocities," by J. Minot Rice and W. Woolsey Johnson, revised edition; New York, 1879.

"An Elementary Treatise on the Differential Calculus, containing the Theory of Plane Curves, with numerous examples," by B. Williamson, M.A., F.R.S., 4th edition, 1880: from the Author.

June 10*th,* 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The following communications were made:—

"On a Binomial Biordinal and the arbitrary Constants of its complete Solution:" Sir James Cockle, F.R.S.

"On the Focal Conics of a Bicircular Quartic:" Mr. H. Hart, M.A.

"Preliminary Note on a Generalization of Pfaff's Problem:" Mr. H. W. Lloyd Tanner, M.A.

"On the Resultant of a Cubic and a Quadric Binary Form:" Prof. Cayley, F.R.S.

"On the Theory of the Focal Distances of Points on Plane Curves:" Mr. W. J. Curran Sharp, M.A.

"Note on the Equation of the two Planes which can be drawn through two given Points to touch a Quadric:" Mr. H. M. Taylor, M.A.

The following presents were received:—

"Educational Times," June, 1880.

"Metaphysische Anfangsgründe der mathematischen Wissenschaften auf Grundlage der heliocentrischen Philosophie dargestellt," von A. Bisharz und P. Dannegger; Sigmaringen, 1880.

"Sur la Détermination d'Intégrales algébriques de Différentielles algébriques," par M. Zeuthen (Comptes Rendus, 10 Mai, 1880).

"Archiv for Mathematik og Naturvidenskab," udgivet af Sophus Lie, W. Müller og G. O. Sars; Fjerde Bind, Fφrste, Andet, Tredie, Fjerde Hefte; Kristiania, 1879.

"Beitrage zur Theorie der Minimalflächen .... (ii.) Metrische Untersuchungen über algebraische Minimalflächen," von Sophus Lie (Math. Annalen, Bd. xv., pp. 465—506).

"Atti della R. Accademia dei Lincei .... Serie Terza.—Memorie della Classe di Scienze Fisiche, Matematiche, e Naturali," Vol. iii., iv.; Roma, 1879.

On a Binomial Biordinal and the Constants of its Complete Solution.

By Sir JAMES COCKLE, M.A., F.R.S., &c.

[*Read June* 10*th*, 1880.]

1. For brevity, I denote differentiations with respect to x, the independent variable, by acute accents. For the sake of enabling the reader to judge of the progress made, I give a brief preliminary application of Boole's canon to the binomial biordinal (see Arts. 2, 3, 4).

2. The symbolical form of

$$(1-x^3)\,z''-\tfrac{3}{2}x^2z'+mxz=0$$

is $D(D-1)\,z-\{D-\frac{11}{4}+\frac{1}{4}\sqrt{1+16m}\}\{D-\frac{11}{4}-\frac{1}{4}\sqrt{1+16m}\}\,x^3z=0$,

which, if we change x^3 into x^2, becomes (see Boole, "Differential Equations," 2nd ed., 1865, p. 430)

$$D(D-\tfrac{2}{3})\,z-\{D-\tfrac{11}{6}+\tfrac{1}{6}\sqrt{1+16m}\}\{D-\tfrac{11}{6}-\tfrac{1}{6}\sqrt{1+16m}\}\,x^2z=0.$$

3. If we put

$$\alpha_1=0,\ \ \alpha_2=\tfrac{2}{3},\ \ \beta_1=\tfrac{11}{6}-\tfrac{1}{6}\sqrt{1+16m},\ \ \beta_2=\tfrac{11}{6}+\tfrac{1}{6}\sqrt{1+16m};$$

then, according to Boole (*ib.*, p. 431), the equation is integrable in finite terms,—1st, if any one of the four quantities represented by $\alpha-\beta$ is an even integer; secondly, if any two of the three quantities

$$\alpha_1-\alpha_2,\ \ \beta_1-\beta_2,\ \ \alpha_1+\alpha_2-\beta_1-\beta_2$$

are odd integers. The integral values are supposed to be either positive or negative, and the even ones to include the value 0.

4. At one time (*ib.*, pp. 466, 467) Boole seems to have thought that these conditions were necessary as well as sufficient. But, however this be, he has not dealt with any cases in which they are not fulfilled. In the case of the above biordinal, the conditions give $m=\frac{1}{4}i\,(i+1)$, where i is an integer positive or negative.

5. Lying outside the range of these conditions are two distinct classes of soluble cases. The above biordinal is soluble if

$$m=(\tfrac{3}{2}i+\tfrac{1}{4})\,(\tfrac{3}{2}i+\tfrac{3}{4}),$$
$$m=\tfrac{5}{64},$$

and there may be other cases of the second class, but I have not as yet determined them. All the forms given in this Article are I believe new, save one which is equivalent to a form cited (and verified) by Professor Cayley (in the "Quarterly Journal of Mathematics," 1879, Vol. xvi., pp. 268—270) from Schwartz (in Crelle-Borchardt, t. 75, 1873; see pp. 292, 326).

6. I obtained the foregoing results by a process of wider application, and which consists in forming, from a given biordinal, a more general equation, and in so transforming the generalized equation as that differentiation shall give rise to a terordinal, whereof the complete solution involves that of the biordinal.

7. Let the given biordinal be

$$z'' + b_1 z' + b_2 z = 0,$$

wherein b_1 and b_2 are functions of x. On this I form the generalized equation

$$z'' + b_1 z' + b_2 z = \frac{ce^{-2\int b_1 dx}}{z^3},$$

c being an arbitrary constant the evanescence of which reduces the latter to the given form. Putting $z = \sqrt{2y}$, we get $z' = \frac{y'}{\sqrt{2y}}$, and $z'' = \frac{y''}{\sqrt{2y}} - \frac{y'^2}{(2y)^{\frac{3}{2}}}$; so that the latter becomes, on multiplying both sides into $(2y)^{\frac{3}{2}}$,

$$2yy'' - (y')^2 + 2b_1 yy' + 4b_2 y^2 = ce^{-2\int b_1 dx},$$

which, differentiated and divided by $2y$, yields

$$y''' + b_1 y'' + b_1 \frac{y'^2}{y} + (b_1' + 4b_2)\, y' + 2b_2' y = -b_1 \left(ce^{-2\int b_1 dx}\right) y^{-1},$$

whence, eliminating $ce^{-2\int b_1 dx}$ and transposing,

$$y''' + 3b_1 y'' + (b_1' + 2b_1^2 + 4b_2)\, y' + 2\,(b_2' + 2b_1 b_2)\, y = 0.$$

8. Of the terordinal, then, $e^{2\int b_1 dx} y$ is an integrating factor; as can be readily verified by differentiating

$$e^{2\int b_1 dx}\left\{yy'' - \tfrac{1}{2}(y')^2 + byy' + 2b_2 y^2\right\} = \tfrac{1}{2}c.$$

From the complete solution of the terordinal will follow that of this first integral, and consequently of the generalized equation, and of the given biordinal.

9. For the biordinal of Art. 2, we have

$$b_1 = -\tfrac{3}{2}x^2 (1 - x^3)^{-1}, \quad b_2 = mx\,(1 - x^3)^{-1},$$

whence, after the proper substitutions, and getting rid of algebraical fractions, we obtain, that which cannot be obtained by mere differentiation, the terordinal

$$(1 - x^3)\, y''' - \tfrac{9}{2}x^2 y'' + (4m - 3)\, xy' + 2my = 0;$$

and when we can completely integrate this terordinal, we can once integrate, and therefore completely integrate, the biordinal of Art. 2.

10. The symbolical form of the terordinal is

$$D\,(D-1)\,(D-2)\, y - x^3\,(D + \tfrac{1}{2})\,(D^2 + D - 4m)\, y = 0;$$

or, which is equivalent,

$$[D]^3\,y-(D-\tfrac{5}{2})\,(D-\tfrac{5}{2}+\tfrac{1}{2}\sqrt{1+16m})\,(D-\tfrac{5}{2}-\tfrac{1}{2}\sqrt{1+16m})\,x^3y=0.$$

11. Take the first class of cases in Art. 5. Here, from

$$m=(\tfrac{3}{2}i+\tfrac{1}{4})\,(\tfrac{3}{2}i+\tfrac{3}{4})$$

we deduce $\quad 1+16m=1+(6i+1)\,(6i+3)=(6i+2)^2.$

Consequently, the symbolical terordinal becomes

$$[D]^3\,y-(D-\tfrac{5}{2})\,(D+3i-\tfrac{5}{2}+1)\,(D-3i-\tfrac{5}{2}-1)\,x^3y=0,$$

which, by the processes of Boole (*op. cit.*, pp. 420, 421), can be transformed into $\quad [D]^3\,v-(D-\tfrac{5}{2})\,(D-\tfrac{5}{2}+1)\,(D-\tfrac{5}{2}-1)\,x^3v=0,$

for i is an integer or 0. But in this case the terordinal may be written

in the form $\quad v-\dfrac{[D-\frac{5}{2}+1]^3}{[D]^3}\,x^3v=0,\;$ or $\;v-\dfrac{[D-\frac{3}{2}]^3}{[D]^3}\,x^3v=0,$

and completely integrated. Its complete solution is

$$v=C_1\sqrt{1-\beta x}+C_2\sqrt{1-\beta^2x}+C_3\sqrt{1-x},$$

where β is an unreal cube root of unity.

12. When $i=0$, then $v=y$. This is the case with the equation discussed by M. Schwartz and Prof. Cayley, which, by a change of the independent variable from x to x^3, and writing z in place of y, is changed

into $$z''-\tfrac{3}{2}\,\frac{x^2}{1-x^3}\,z'+\frac{\frac{3}{16}x}{1-x^3}\,z=0.$$

One form of the complete solution of the terordinal will be obtained by making $C_3=0$; and $C_1\sqrt{1-\beta x}$ and $C_2\sqrt{1-\beta^2x}$ are incomplete solutions, which may be written $\sqrt{C_1^2-\beta C_1^2x}$ and $\sqrt{C_2^2-\beta^2C_2^2x}$. Let α be a prime sixth root of negative unity. All the four equations $C_1^2=\alpha$, $\beta C_1^2=\alpha^5$, $C_2^2=-\alpha^5$, $-\beta^2C_2^2=\alpha$ can be satisfied; for the elimination of C_1 gives $\beta=\alpha^4$, and that of C_2 gives $\beta^2=\alpha^{-4}=\alpha^{-6}\alpha^2=-\alpha^2$; and hence we get $\beta+\beta^2=-1=\alpha^4-\alpha^2$, which is right. Now a first integral of the terordinal, as obtained by the factor, is

$$(1-x^3)\,yy''-\tfrac{1}{2}\,(1-x^3)\,y'^2-\tfrac{3}{2}x^2yy'+\tfrac{3}{8}xy^2=\tfrac{1}{2}c,$$

and the complete solution of this first integral is

$$y=\dot{C}_1\sqrt{1-\beta x}+\dot{C}_2\sqrt{1-\beta^2x}+\dot{C}_3\sqrt{1-x},$$

wherein $\dot{C}_1$, $\dot{C}_2$, and $\dot{C}_3$ are connected by the relation

$$\dot{C}_3^2+\beta^2\dot{C}_1^2+\beta\dot{C}_2^2=-\frac{8}{3}\,\frac{c}{2}.$$

Let $c = 0$ and $\dot{C}_3 = 0$, then $\dot{C}_2^2 : \dot{C}_1^2 = -\beta$ follows. But

$$(-a)^{\frac{5}{2}} : a^{\frac{1}{2}} = \sqrt{-1}\, a^2 = \sqrt{-1}\,(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}) = \sqrt{-1}\,\beta_n;$$

or $$C_2 : C_1 = \sqrt{-1}\,\beta_n, \text{ and } C_2^2 : C_1^2 = -\beta_n^2,$$

where β_n represents either value of β, as indeed does the unsuffixed β which precedes it. Hence, since $C_2 : C_1$ or $\dot{C}_2 : \dot{C}_1$ takes the double sign, and the Y of Professor Cayley is the same thing as my y, it appears, from either investigation, that when c vanishes the equation of the second order has the two particular integrals

$$\sqrt{(a - a^5 x)} \pm \sqrt{(-a^5 + ax)}.$$

13. Take the class of cases (Boole's) adverted to in Arts. 3 and 4. Here, from $m = \frac{1}{4} i\,(i+1)$, we deduce

$$1 + 16m = 4i^2 + 4i + 1 = (2i+1)^2.$$

Consequently, the symbolical terordinal becomes

$$[D]^3 y - (D - \tfrac{5}{2})(D - \tfrac{5}{2} + i + \tfrac{1}{2})(D - \tfrac{5}{2} - i - \tfrac{1}{2})\, x^3 y = 0;$$

or, in other words, it takes one of the two forms

$$[D]^3 y - (D - \tfrac{5}{2})(D + i - 2)(D - i - 3)\, x^3 y = 0,$$
$$[D]^3 y - (D - \tfrac{5}{2})(D + i - 3)(D - i - 2)\, x^3 y = 0;$$

i.e., one or other of the forms

$$y - \frac{(D - \frac{5}{2})(D + i - 2)(D - i - 3)}{D(D-1)(D-2)}\, x^3 y = 0,$$
$$y - \frac{(D - \frac{5}{2})(D + i - 3)(D - i - 2)}{D(D-1)(D-2)}\, x^3 y = 0.$$

If $i \equiv 0$ (mod. 3), these forms respectively may, by means of the algorithm of Boole, be made to coalesce in

$$v - \frac{D - \frac{5}{2}}{D - 1}\, x^3 v = 0;$$

and if $i \equiv 1$ (mod. 3), they may, by that algorithm, be respectively transformed into

$$v - \frac{(D - \frac{5}{2})(D-1)}{D(D-2)}\, x^3 v = 0, \text{ and } v - \frac{(D - \frac{5}{2})}{D - 1}\, x^3 v = 0,$$

the former of which is soluble by Boole's process, and the latter is a primordinal. If $i \equiv 2$ (mod. 3), we merely reverse the order of these last results. The constants of the ultimate solutions must be supplied in the way pointed out by Boole (*op. cit.*, p. 421).

14. Thus far the algorithm of Boole has sufficed for the discussion of the foregoing new transformations. It now becomes necessary to

have recourse to the results of Mr. Harley. The formulæ in the next Article are taken from page 348 of Mr. Harley's paper "On the Theory of the Transcendental Solution of Algebraic Equations" in the "Quarterly Journal of Mathematics" (Vol. v., 1862), and also from page 237 of another paper of Mr. Harley ("On a Certain Class of Linear Differential Equations," in the "Memoirs of the Manchester Society," 3rd Series, Vol. ii., 1865).

15. If, in the quartic $y^4-4y+3x=0$,

we regard y as the dependent, and x as the independent variable, we have the following results :

$$4\,(1-x^3)\,\frac{dy}{dx}=-\,(y^3+xy^2+x^2y-3),$$

$$4^2\,(1-x^3)^2\,\frac{d^2y}{dx^2}=-\,\{6x^2y^3+(1+5x^3)\,y^2+3\,(1+x^3)\,xy-18x^2\},$$

$$2^5\,(1-x^3)\,\frac{d^3y}{dx^3}-2^4.3^2x^2\,\frac{d^2y}{dx^2}-2.43x\,\frac{dy}{dx}+5y=0,$$

this last equation being the differential resolvent of the quartic.

16. I found that

$$2^5\,(1-x^3)\,y\,\frac{d^2y}{dx^2}-2^4\,(1-x^3)\left(\frac{dy}{dx}\right)^2-3\,.\,2^4x^2y\,\frac{dy}{dx}+5xy^2=16c$$

is a first integral of this resolvent. On my sending this integral to Mr. Harley, he soon saw that the substitution, for y, of a root of the quartic would or ought to reduce the sinister of the integral to a constant. This constant Mr. Harley determined as follows, or substantially as follows.

17. He found, after reductions where necessary,

$$4^2\,(1-x^3)^2\,yy''=-\,(1+5x^3)\,y^3-3\,(1+x^3)\,xy^2-6x^2y+18x^3,$$

$$4^2\,(1-x^3)^2\,(y')^2=-\,(2y^3+xy^2-9)\,(1-x^3),$$

$$4\,(1-x^3)\,yy'=-\,xy^3-x^2y^2-y+3x.$$

Now, multiplying the sinister of the integral into $(1-x^3)$, and substituting, we arrive at

$$-9\,(1-x^3)=16c\,(1-x^3),\ \text{or}\ c=-\frac{9}{16}=-\frac{3^2}{4^2}.$$

18. It follows at once from Mr. Harley's evaluation, and from the fact that the sinister of the integral is homogeneous in y, that, if y_r be any root of the quartic, then the expression

$$\tfrac{4}{3}y_r\sqrt{-c}$$

yields four particular integrals of the first integral.

19. Following up the path thus entered by Mr. Harley when he reduced the sinister to a constant, and calling to mind that, by establishing the proper relation between the constants, the complete solution of the resolvent may be made to furnish that of its first integral, I assume $y = a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4$, or $y = \Sigma a y$, wherein y_r is a root of the quartic and a_r constant. In virtue of the relation $\Sigma y = 0$, given by the quartic, we may, if we please, suppose any one of the a's to vanish. On substituting this value of y in the sinister of the first integral, we get [having first multiplied the integral into $(1-x^3)$ for convenience]

$$\begin{aligned}&\Sigma a^2 \{2^5(1-x^3)^2 yy''-2^4(1-x^3)^2(y')^2-3.2^4(1-x^3)x^2yy'+5(1-x^3)xy^2\}\\&+\Sigma a_1 a_2 \{2^5 (1-x^3)^2 (y_1 y_2''+y_2 y_1'')\\&\qquad -2^5(1-x^3)^2 y_1' y_2' - 3 \,.\, 2^4 (1-x^3)\, x^2 (y_1 y_2' + y_2 y_1')\\&\qquad +10(1-x^3)\, x y_1 y_2\} = 16c\,(1-x^3).\end{aligned}$$

20. But, in virtue of Mr. Harley's result, this equation may be put under the form

$$\begin{aligned}&\Sigma a_1 a_2 \{2^4 (1-x^3)^2 (y_1 y_2''+y_2 y_1'')-2^4(1-x^3)^2 y_1' y_2'\\&\qquad -3\,.\,2^3(1-x^3)\,x^2(y_1y_2'+y_2y_1')+5\,(1-x^3)\,xy_1y_2\}\\&\qquad\qquad = \{8c+\tfrac{9}{2}\Sigma a^2\}\,(1-x^3).\end{aligned}$$

21. To avoid suffixes, let us for a moment denote by y and z any two roots of the quartic. Then $y^4-z^4-4\,(y-z)$ vanishes. Hence we get

$$y^3+z^3+yz\,(z+y) = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

which, multiplied into z^3 and reduced by relations derived from the quartic, gives $y^3z^3-3x\,(y^2+z^2)+4yz\,(y+z)-3xyz=0$ ....(2);

again, (1) multiplied into y and reduced gives

$$yz\,(y^2+yz+z^2) = 3x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

also, (3) multiplied into y and reduced gives

$$y^3z^2+z^3y^2+4yz-3x\,(y+z)=0 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

22. Now, remembering that Mr. Harley's formulæ, given in Art. 15, remain true when y is therein replaced by z, we find

$$\begin{aligned}&4^2(1-x^3)^2\,(yz''+zy'') = -\,6x^2\,(y^2+z^2)\,yz\\&\qquad -(1+5x^3)\,yz\,(y+z)-6x\,(1+x^3)\,yz+18x^2\,(y+z);\end{aligned}$$

also, after eliminating y^3z^3 by means of (2), and $y^3z^2+z^3y^2$ by means of (4), and using (3) and reducing, we get

$$4^2\,(1-x^3)^2\,y'z' = -\,(1-x^3)\,\{3+(x+y+z)\,yz\};$$

we also obtain

$$4\,(1-x^3)\,(yz'+zy') = -\,yz\,(y^2+z^2)-xyz\,(y+z)-2x^2yz+3\,(y+z).$$

23. Replacing y by y_1, and z by y_2, and substituting accordingly in the formulæ of Art. 20, we find that the sinister of the equation therein given becomes constant, and the relation itself becomes

$$\Sigma a_1 a_2 \{3(1-x^3)\} = \{8c + \tfrac{9}{2}\Sigma a^2\}(1-x^3),$$

whence $$6\Sigma a_1 a_2 - 9\Sigma a^2 = 16c;$$

and when $c=0$, we have $2\Sigma a_1 a_2 - 3\Sigma a^2 = 0$, or

$$(\Sigma a)^2 = 4\Sigma a^2 \quad\text{..............................(5).}$$

24. But the differential resolvent of the above quartic is the terordinal to which, by the process of Art. 7, we are led from

$$z'' - \tfrac{3}{2}\,\frac{x^2}{1-x^3}\,\frac{dz}{dx} + \frac{\frac{5}{64}x}{1-x^3}\,z = 0 \quad\text{..................(6).}$$

The complete solution of (6) is therefore

$$z = \sqrt{2\Sigma ay},$$

the constants, which, in consequence of Σy vanishing, are effectively three only, being connected by (5). And, since (5) only determines ratios, we may write the solution thus

$$z = \sqrt{\Sigma ay} \quad\text{..............................(7),}$$

the a's being connected by (5). This is a complete solution of (6). But, (6) being linear, the sum of any number of its independent solutions is a solution. Consequently

$$z = \sqrt{\Sigma ay} + \sqrt{\Sigma \beta y} + \sqrt{\Sigma \gamma y} + \ldots,$$

or, say $$z = S\sqrt{\Sigma \beta y} \quad\text{..............................(8),}$$

the number of terms included under the summation S being taken as infinite if we please, is a solution. And all biordinals soluble by means of the terordinal will have their solutions of the form $z = S\sqrt{\Sigma\beta y}$, the summation Σ extending to three independent particular integrals of the terordinal. The biordinal of Schwartz (and Cayley, in whose paper the first $-$ should be $+$) is connected with hypergeometric series; that of (6), wherein $m = \frac{5}{64}$, with ultra-hypergeometric series (see "Manchester Proceedings," Session 1862–3, pp. 19, 20).

25. Consider the more general binomial biordinal

$$z'' - \left(n - 1 + \frac{\frac{3}{2}}{1-x^3}\right)\frac{1}{x}\,z' - \frac{m}{x^2}\,z = 0.$$

The auxiliary terordinal (Art. 7) is

$$y''' - 3\left(n-1+\frac{\frac{3}{2}}{1-x^3}\right)\frac{1}{x}\,y''$$
$$+\left(2n^2 - 3n + 1 - 4m + \frac{6n}{1-x^3}\right)\frac{1}{x^2}\,y' + 4m\left(n + \frac{\frac{3}{2}}{1-x^3}\right)\frac{1}{x^3}\,y = 0,$$

or $$(1-x^3)\,x^3y'''-3\left\{n+\tfrac{1}{2}-(n-1)\,x^3\right\}x^2y''$$
$$+\left\{2n^2+3n+1-4m-(2n^2-3n+1-4m)\,x^3\right\}xy'+4m\,(n+\tfrac{3}{2}-nx^3)\,y=0.$$

26. This terordinal may be put under the comparatively simple symbolical form

$$y-\frac{\{D-(n+3)\}\left[\{D-(n+3)\}^2-\{n^2+4m\}\right]}{(D-N)\left\{(D-N)^2-(N^2+4m)\right\}}\,x^3y=0,$$

where $N\equiv n+\frac{3}{2}$, which may be still further simplified. For if, by Boole's algorithm, we change the D in the numerator to $D+3$, we get a transformed equation, which may be written

$$v-\frac{(D-n)\left\{(D-n)^2-(n^2+4m)\right\}}{(D-N)\left\{(D-N)^2-(N^2+4m)\right\}}\,x^3v=0,$$

or $$v-\frac{(D-n)(D-n+\sqrt{n^2+4m})(D-n-\sqrt{n^2+4m})}{(D-N)(D-N+\sqrt{N^2+4m})(D-N-\sqrt{N^2+4m})}\,x^3v=0.$$

27. From this last auxiliary terordinal we can construct soluble binomial biordinals, distinct alike from any integrable by Boole's canons and from the new forms solved earlier in this paper. In proof of this proposition I shall deduce one such biordinal. There may be, and probably are, others, but I have not as yet determined them.

28. Suppose that $N^2+4m=L^2$, $n^2+4m=l^2$, where L and l are rational. Then the terordinal becomes

$$v-\frac{(D-n)\,(D-n+l)\,(D-n-l)}{(D-N)\,(D-N+L)\,(D-N-L)}\,x^3v=0.$$

29. Let $$L\equiv l\equiv\pm1 \pmod{3},$$

then the terordinal may be made to take the form

$$v-\frac{[D-n+1]^3}{[D-N+1]^3}\,x^3v=0,$$

which breaks up into primordinals.

30. Now the congruences $L\equiv\pm1 \pmod{3}$ and $l\equiv\pm1 \pmod{3}$ may be written as the equations

$$N^2+4m=(3j\pm1)^2,\quad n^2+4m=(3i\pm1)^2,$$

j and i being any integers, positive or negative. Subtracting the latter from the former, and remembering that $N=n+\frac{3}{2}$, we get

$$3n+\frac{3^2}{2^2}=3^2\,(j^2-i^2)\pm6\,(j-i),$$

whence $$n=(j-i)\,(3j+3i\pm2)-\tfrac{3}{4};$$

and m will be formed by substitution in

$$m=\tfrac{1}{4}\{(3i\pm 1)^2-n^2\}.$$

31. Hence we easily get a form hitherto unsolved, and which would perhaps heretofore have been deemed insoluble. Let i and j both vanish, as they may. Then

$$n=-\tfrac{3}{4},\quad m=\tfrac{1}{4}\left(1-\tfrac{9}{16}\right)=\tfrac{7}{64},$$

and the binomial biordinal of Art. 25 becomes

$$z''+\left(\tfrac{7}{4}-\frac{\frac{3}{2}}{1-x^3}\right)\frac{1}{x}\,z'-\tfrac{7}{64}\,\frac{1}{x^2}\,z=0,$$

which takes the symbolical form

$$(D+\tfrac{1}{8})\,(D-\tfrac{7}{8})\,z-(D-3+\tfrac{7}{8})\,(D-3-\tfrac{1}{8})\,x^3z\,;$$

or, changing x^3 into x^2,

$$(D+\tfrac{1}{12})\,(D-\tfrac{7}{12})\,z-(D-2+\tfrac{7}{12})\,(D-2-\tfrac{1}{12})\,x^2z=0,$$

a form not soluble through Boole's canon, though admitting of a complete solution by means of the auxiliary terordinal.

32. The particular biordinals and terordinals above considered are of the form called, by Boole, binomial. But such a form is not essential to the applicability of the process described in Arts. 7 and 8, though it may facilitate the application. It is, however, a matter of quite fundamental importance to ascertain what forms of binomial differential equations admit of finite solution (Boole, *Phil. Trans.* for 1864, p. 733). It may be that an auxiliary quartordinal, or higher ordinal, can be used as a resolvent of a biordinal.

Preliminary Note on a Generalization of Pfaff's Problem.

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1. Pfaff's problem relates to the transformation of an expression

$$y_1dx_1+y_2dx_2+\ldots+y_ndx_n$$

into another expression of similar form

$$v_1du_1+v_2du_2+\ldots+v_rdu_r,$$

the y's, v's, and u's being functions of the x's; and in connection with this arises the question as to the necessary and sufficient conditions that the transformation may be possible for any proposed value of r.

The present note deals with the transformation of a system of m expressions

$$y_{i1}dx_1+y_{i2}dx_2+\ldots+y_{in}dx_n\ (=Y_i),$$
$$i=1,\,2,\,\ldots\,m,$$

into a system of similar form

$$v_{i1}du_1+v_{i2}du_2+\ldots+v_{ir}du_r\ (=V_i),$$
$$i=1,\,2,\,\ldots\,m,$$

viz.,

$$Y_i=V_i.$$

As before, the y's, u's, and v's are functions of the x's. The chief object of the note is to present the necessary and sufficient conditions for the possibility of the transformation for each value of r; or, what is the same thing, the determination of the smallest value of r which belongs to a given system, Y_i.

This problem has, in one case at any rate, attracted considerable attention. When $r=m$ the equations

$$Y_i=0$$

form an "integrable" system, the solutions being

$$u_i=c_i.$$

The conditions that $Y_i=0$ should be an integrable system ought therefore to be included in those hereinafter given, and it will be seen that this is the case.

2. The results obtained depend upon a theorem enunciated below. They are similar to, and include, those given in two papers "On the Transformation of a Linear Differential Expression," published in the "Quarterly Journal" (Vol. XVI., pp. 45—64).

The theorem asserts the equivalence of two systems of equations, the form of which we proceed to indicate.

Write down the m rows of y's, and under these a number, say k, of pairs of rows; the upper row of each pair being the operative symbols

$$\frac{d}{dx_1},\ \frac{d}{dx_2},\ \cdots\cdots\ \frac{d}{dx_n},$$

the lower row being any row of y's. These are to be arranged so that the second subscript of every y, and the subscript of every $\frac{d}{dx}$, in a vertical column, may be the same. From the system thus written, form determinants containing $m+2k$ columns each; and in developing these it is assumed that the p^{th} "factor" of each term is taken from the p^{th} row. The equations which express that all these determinants vanish, make up the first system. They may be written

$$\left\|\begin{array}{cccc} y_{11}, & y_{12}, & \cdots & y_{1n} \\ \cdots & \cdots & & \cdots \\ y_{m1}, & y_{m2}, & \cdots & y_{mn} \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \cdots & \dfrac{d}{dx_n} \\ y_{i1}, & y_{i2}, & \cdots & y_{in} \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \cdots & \dfrac{d}{dx_n} \\ y_{j1}, & y_{j2}, & \cdots & y_{jn} \\ \cdots & \cdots & & \cdots \end{array}\right\| = 0 \ldots\ldots\ldots\ldots\text{(I)},$$

$$i, j, \ldots = 1, 2, \ldots m.$$

A more convenient form for many purposes is got by using the umbral notation; viz., we write

$$\left\|\begin{array}{l} 1, 2, \ \ldots\ldots\ldots\ldots \ \ n \\ 1, 2, \ldots m, \delta, i, \delta, j \ldots \end{array}\right\| = 0,$$

where the upper row consists of column-subscripts; that is, of the second subscripts of the y's, and the subscripts of the $\dfrac{d}{dx}$'s. The lower row contains the first subscripts of the y's, and each row of $\dfrac{d}{dx}$ is represented by a δ.

The second system of equations is

$$\left\|\begin{array}{cccc} y_{i1}, & y_{i2}, & \cdots & y_{in} \\ \dfrac{du_1}{dx_1}, & \dfrac{du_1}{dx_2}, & \cdots & \dfrac{du_1}{dx_n} \\ \cdots & \cdots & & \cdots \\ \dfrac{du_r}{dx_1}, & \dfrac{du_r}{dx_2}, & \cdots & \dfrac{du_r}{dx_n} \end{array}\right\| = 0 \ \ldots\ldots\ldots\ldots\text{(II.)},$$

$$i = 1, 2, \ldots m.$$

The theorem is, that when

$$r+1 = m+k \ \ldots\ldots\ldots\ldots\text{(III.)},$$

the two systems (I.), (II.) are equivalent; that is to say, supposing (I.) to be satisfied, values of $u_1, \ldots u_r$ can be found to satisfy (II.); and, given (II.), the system (I.) becomes an identity, whatever $u_1, \ldots u_r$ may be.

The application of this to the problem of Art. 1, is obvious. From (II.), we have at once

$$y_{ij} = v_{i1}\frac{du_1}{dx_j} + v_{i2}\frac{du_2}{dx_j} + \ldots + v_{ir}\frac{du_r}{dx_j}.$$

This gives

$$y_{i1}dx_1 + y_{i2}dx_2 + \ldots + y_{in}dx_n = v_{i1}du_1 + v_{i2}du_2 + \ldots + v_{ir}du_r,$$

or

$$Y_i = V_i.$$

Thus the conditions that the proposed transformation should be possible are the equations (I.) And, supposing these to be satisfied when there are k double rows, but not when there are $k-1$ of these rows, the smallest value of r is given by (III.).

3. The proof of the second part of the theorem is very simple. When (II.) is satisfied, we have

$$y_{ij} = v_{i1}\frac{du_1}{dx_j} + \ldots + v_{ir}\frac{du_r}{dx_j}.$$

Substitute in (I.): and observe that every minor formed from a double row becomes a sum of Jacobians of u, v; for instance,

$$\begin{vmatrix} \dfrac{d}{dx_1}, & \dfrac{d}{dx_2} \\ y_{i1}, & y_{i2} \end{vmatrix} = \frac{d\,(v_{i1},\,u_1)}{d\,(x_1,\,x_2)} + \frac{d\,(v_{i2},\,u_2)}{d\,(x_1,\,x_2)} + \&c.$$

The y's are linear functions of $\dfrac{du}{dx}$. Hence, whatever r may be, each determinant of (I.) becomes, after the substitution is made, a linear function of Jacobians, such as

$$\frac{d\,(u_1,\,\ldots\,u_m,\,v_{ip},\,u_p,\,v_{jq},\,u_q,\,\ldots)}{d\,(x_1,\,x_2,\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots)}.$$

In the numerator there are k v's, and $\overline{m+k}$ u's. But there are only r different u's, so that, if $r = m+k-1$ ....(III.), two u's must be identical, and the Jacobian vanishes identically. Hence the truth of (I.) is manifest. Otherwise expressed, the system (I.) are *necessary* conditions of the possibility of the transformation of Art. 1.

4. Of the first part of the theorem I have not yet found a satisfactory proof, and it is mainly in the hope of having this defect removed that I have laid the results before the Society. There is not much difficulty in verifying this part of the theorem, though the reductions are somewhat formidable. I have, in fact, succeeded in this attempt, following the general line adopted in my papers above cited. But it is hardly worth while giving this work, as, although it shows the truth of the assertion, it gives no reason for the peculiar form of the equations involved. Instead of giving this verification, I propose to indicate briefly some remarkable properties of the equations (I.) upon which a real proof may probably be based, or which may at least lead to a more concise verification.

5. The action of the operative symbols, $\dfrac{d}{dx}$, in any row, does not ex-

tend beyond the next following row. Imagine any determinant of (II.) expanded as a sum of terms, such as

$$A\begin{vmatrix} \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \dfrac{d}{dx_3} \\ y_1, & y_2, & y_3 \\ \dfrac{d}{dx}, & \dfrac{d}{dx_2}, & \dfrac{d}{dx_3} \end{vmatrix}B,$$

where A is a minor determinant formed from the upper rows; B is formed from the lower rows; and in A, B, and the minor written at length, all the rows of the original determinant are included. Developing the middle minor, it will be seen that we get no terms involving differential coefficients of the second order; or, more truly, such terms occur in pairs with opposite signs. Thus we have two terms

$$A\frac{d}{dx_1}y_2\frac{d}{dx_3}B - A\frac{d}{dx_3}y_2\frac{d}{dx_1}B,$$

which give $A\dfrac{dy_2}{dx_1}\cdot\dfrac{dB}{dx_3}+Ay_2\dfrac{d^2B}{dx_1\,dx_2}-A\dfrac{dy_2}{dx_3}\cdot\dfrac{dB}{dx_1}-Ay_2\dfrac{d^2B}{dx_3\,dx_1}.$

Hence it follows that the k double rows in (I.) may be transposed *ad libitum.*

6. The equation (I.) is not affected when any row of y's is multiplied by any quantity, say λ. Let the row of y's to be multiplied, be $y_{11}, y_{12}, \dots y_{1n}$, and bring the double row containing the changed y's to the top of the double rows. Then, observing that

$$\frac{d}{dx}\lambda y = \lambda\frac{d}{dx}\cdot y + y\frac{d\lambda}{dx},$$

we have
$$\begin{vmatrix} y_{11}, & \cdots \\ \cdots & \cdots \\ \dfrac{d}{dx_1}, & \cdots \\ \lambda y_{11}, & \cdots \\ \cdots & \cdots \end{vmatrix} = \begin{vmatrix} y_{11}, & \cdots \\ \cdots & \cdots \\ \lambda\dfrac{d}{dx_1}, & \cdots \\ y_{11}, & \cdots \\ \cdots & \cdots \end{vmatrix} + \begin{vmatrix} y_{11}, & \cdots \\ \cdots & \cdots \\ y_{11}, & \cdots \\ \dfrac{d\lambda}{dx_1}, & \cdots \\ \cdots & \cdots \end{vmatrix}$$

whereof the latter determinant vanishes identically. Thus the effect of multiplying a row of y's by λ, is merely to multiply the expression on the left of (I.) by λ, so that the truth of (I.) is unaffected.

Hence, it follows, that if the system (I.) be satisfied for m expressions $Y_1, \dots Y_m$, it is also satisfied for any set of m independent linear combinations of $Y_1, \dots Y_m$.

7. A remarkable property of the determinants upon the left of (I.)

is, that if the order of the rows be reversed, the determinants change in sign, or not at all.

Take a particular example:—We have

$$\frac{d}{dx_2}\cdot y_{23}y_{14} = y_{23}\frac{d}{dx_2}y_{14} + y_{14}\frac{d}{dx_2}y_{23},$$

and the like. Hence

$$\begin{vmatrix} 1, 2, 3, 4 \\ 1, \delta, 2, 1 \end{vmatrix} = \begin{vmatrix} 1, 3, 2, 4 \\ 1, 2, \delta, 1 \end{vmatrix} + \begin{vmatrix} 1, 4, 2, 3 \\ 1, 1, \delta, 2 \end{vmatrix}.$$

Of these the second determinant vanishes, since it contains two identical rows unseparated by a row of operators. The first determinant on the right

$$= -\begin{vmatrix} 1, 2, 3, 4 \\ 1, 2, \delta, 1 \end{vmatrix},$$

and in this the order of the rows is the reverse of the original order. The proof is quite general in its character; and it shows that, in the equations (I.), the order of the rows may be reversed.

We are thus led to sets of equations intermediate between those given in (I.), consisting of $m+2k-1$ rows instead of $m+2k$. For instance, corresponding to (I.), we have

$$\begin{vmatrix} 1, 2, \ldots\ldots\ldots\ldots \\ i, \delta, j, \delta, 1, 2, \ldots m \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV.)},$$

consisting of $\overline{m+4}$ rows. But there is another set of equations

$$\begin{vmatrix} 1, 2, \ldots\ldots\ldots\ldots \\ \delta, j, \delta, 1, 2, \ldots m \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(V.)},$$

consisting of $\overline{m+3}$ rows, and not included in (I.). The form of (II.) corresponding to (V.) is the same as that belonging to (IV.), but there is the additional circumstance that one of the determinants

$$|\, v_{ij} \,|$$

can be made constant, which is not the case if (IV.) be satisfied, but not (V.). This corresponds to what is known to happen in the case of $m = 1$.

If the order of the rows in (V.) be reversed, we get a symbol of operation which is continually appearing in the verification to which allusion has been made above. If the order of the last $m+2$ rows be reversed, the equation is unaltered.

8. The inversion of the last article leads to a tolerably easy method of partially solving the problem in any particular case. For simplicity, we shall consider a special case $m = 2$, $k = 1$; and, moreover, we will suppose $n = m+2k = 4$; so that, each matrix in (I.) consists of one

determinant only. We have, then, the equations

$$\begin{vmatrix} y_{11}, & \dots & y_{14} \\ \frac{d}{dx_1}, & \dots & \dots \\ y_{11}, & \dots & \dots \\ y_{21}, & \dots & \dots \end{vmatrix} = 0, \quad \begin{vmatrix} y_{21}, & \dots & y_{24} \\ \frac{d}{dx_1}, & \dots & \dots \\ y_{11}, & \dots & \dots \\ y_{21}, & \dots & \dots \end{vmatrix} = 0;$$

but the first minors do not vanish. Let these first minors, formed by erasing the top row, and omitting in turn the 1st, 2nd, 3rd, 4th columns, be called $\eta_1, \eta_2, \eta_3, \eta_4$. Then we have

$$y_{11}\eta_1 + y_{12}\eta_2 + \dots = 0,$$
$$y_{21}\eta_1 + y_{22}\eta_2 + \dots = 0,$$

and
$$\frac{d\eta_1}{dx_1} + \frac{d\eta_2}{dx_2} + \frac{d\eta_3}{dx_3} + \frac{d\eta_4}{dx_4} = 0.$$

From the latter equation ("Messenger of Mathematics," Vol. vii., pp. 107, &c.), we have

$$\eta_1, -\eta_2, \eta_3, -\eta_4 = \frac{d(\xi_1, \xi_2, \xi_3)}{d(x_1, x_2, x_3, x_4)},$$

and then the first two equations become

$$\begin{vmatrix} y_{i1}, & y_{i2}, & \dots \\ \frac{d\xi_1}{dx_1}, & \dots & \dots \\ \frac{d\xi_2}{dx_1}, & \dots & \dots \\ \frac{d\xi_3}{dx_1}, & \dots & \dots \end{vmatrix} = 0$$
$$(i = 1, 2).$$

The ξ's are found by a process indicated in the note above quoted, and give the values of u_1, u_2, and $\begin{vmatrix} v_{11}, & v_{12} \\ v_{21}, & v_{22} \end{vmatrix}$.

When n is greater than $m+2k$, we have more than one differential equation, and the η's have more than one subscript each. By treating these equations as described in a paper of mine, in the "Proceedings" of the Society (Vol. x., pp. 55, &c.), we get precisely similar results. Starting with (I.), containing $m+2k$ rows, we can express the y's in terms of $m+2k-1$ functions ξ_1, of which $m+k-1$ agree with the u's and the rest depend upon the v's, if a proper choice be made. In such cases we may form more than one set of ξ's; and then our work is facilitated, since only the ξ's common to all the sets can be u's. To see how these different sets arise, consider the case of $k=2$. We have a

set of equations
$$\begin{vmatrix} y_{i1}, & \dots \\ \dfrac{d}{dx_1}, & \dots \\ y_{11}, & \dots \\ \dfrac{d}{dx_1}, & \dots \\ \vdots & \end{vmatrix} = 0,$$
where $i = 1, 2,$ &c. These give one set of first minors. Then there is another set
$$\begin{vmatrix} y_{i1}, & \dots \\ \dfrac{d}{dx_1}, & \dots \\ y_{21}, & \dots \\ \dfrac{d}{dx_1}, & \dots \\ \vdots & \end{vmatrix} = 0,$$
which gives a different set of first minors, and consequently a different set of ξ's. And so on.

Precisely the same process applies to the equations of Art. 7, with $m+2k-1$ rows.

9. It may be well to show that the conditions above written include, as a particular case, the known conditions for $r = m$ (so that $k = 1$). It is convenient to express these conditions in the form given by Boole, and it will only be necessary to take a particular example.

Let
$$\Delta_1 = \frac{d}{dx_1} - y_{13}\frac{d}{dx_3} - y_{14}\frac{d}{dx_4},$$
$$\Delta_2 = \frac{d}{dx_2} - y_{23}\frac{d}{dx_3} - y_{24}\frac{d}{dx_4};$$
then the necessary and sufficient conditions that
$$y_{13}\,dx_1 + y_{23}\,dx_2 + dx_3,$$
$$y_{14}\,dx_1 + y_{24}\,dx_2 + dx_4$$
should be an integrable system, are
$$\Delta_1 y_{13} = \Delta_2 y_{23}, \quad \Delta_1 y_{14} = \Delta_2 y_{24}.$$
The conditions (I.) are
$$\begin{vmatrix} y_{13}, & y_{23}, & 1, & 0 \\ y_{14}, & y_{24}, & 0, & 1 \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \dfrac{d}{dx_8}, & \dfrac{d}{dx_4} \\ y_{13}, & y_{23}, & 1, & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} y_{13}, & y_{23}, & 1, & 0 \\ y_{14}, & y_{24}, & 0, & 1 \\ \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \dfrac{d}{dx_3}, & \dfrac{d}{dx_4} \\ y_{14}, & y_{24}, & 0, & 1 \end{vmatrix} = 0,$$

which, upon development, are found to be identical with those written above.

10. If, in (I.), we take $2k = n - m + 1$,

the number of columns is less than the number of rows, and no condition is required to be satisfied by the y's in order that the corresponding transformation may be possible. The value of r in this case is $\frac{1}{2}(m+n-1)$; and therefore, whatever the y's may be, the system Y_i can be changed into a system V_i with $\frac{1}{2}(m+n-1)$ variables u.

Should $n-m$ be even, we may put

$$2k = n - m + 2;$$

and the system Y can be changed into a system V with $\frac{1}{2}(m+n)$ variables u. In this case we are dealing with equations of condition like (V.), and therefore it is possible, in the system V, to make one of the determinants $|v_{ij}|$ constant, or, say, equal to unity.

By putting $m = 1$ we reproduce the known results for a single expression Y.

On a Formula of Elimination. *By* Prof. A. Cayley, F.R.S.

[*Read June* 10*th*, 1880.]

Consider the equations

$$(a, \ldots \between \theta, 1)^n = 0,$$
$$(A, \ldots \between \theta, 1)^m = 0,$$

where $a, \ldots, A, \ldots$ are functions of coordinates. To fix the ideas, suppose that each of these coefficients is a linear function of the four coordinates x, y, z, w. Then, eliminating θ, we obtain $\nabla = 0$, the equation of a surface; and (as is known) this surface has a nodal curve.

It is easy to obtain the equations of the nodal curve in the case where one of the equations, say the second, is a quadric: the process is substantially the same, whatever may be the order of the other equation, and I take it to be a cubic; the two equations therefore are

$$(a, b, c, d \between \theta, 1)^3 = 0,$$
$$(A, B, C \between \theta, 1)^2 = 0;$$

giving rise to an equation

$$\nabla, = (a, b, c, d)^2 (A, B, C)^3, = 0.$$

And it is required to perform the elimination so as to put in evidence the nodal line of this surface.

Take θ_1, θ_2 the roots of the second equation, or write

$$(A, B, C \between \theta, 1)^2 = A(\theta-\theta_1)(\theta-\theta_2);$$

that is
$$\theta_1+\theta_2 = -\frac{2B}{A}, \quad \theta_1\theta_2 = \frac{C}{A};$$

then, if
$$\Theta_1 = (a, b, c, d \between \theta_1, 1)^3,$$
$$\Theta_2 = (a, b, c, d \between \theta_2, 1)^3,$$

we have
$$\nabla = A^3\Theta_1\Theta_2;$$

viz., on the right-hand side, replacing the symmetrical functions of θ_1, θ_2 by their values in terms of A, B, C, we have the expression of ∇ in its known form
$$\nabla = a^2C^3 + \&c.$$

Form now the expressions $\Theta_1-\Theta_2$, $\theta_2\Theta_1-\theta_1\Theta_2$, $\theta_2^2\Theta_1-\theta_1^2\Theta_2$, $\theta_2^3\Theta_1-\theta_1^3\Theta_2$, each divided by $\theta_1-\theta_2$. These are evidently symmetrical functions of θ_1, θ_2, the values being given by the successive lines of the expression

$$\left(\begin{array}{cccc} 0, & 1, & \theta_1+\theta_2, & \theta_1^2+\theta_1\theta_2+\theta_2^2 \\ -1, & 0, & \theta_1\theta_2, & \theta_1\theta_2(\theta_1+\theta_2) \\ -(\theta_1+\theta_2), & -\theta_1\theta_2, & 0, & \theta_1^2\theta_2^2 \\ -(\theta_1^2+\theta_1\theta_2+\theta_2^2), & -\theta_1\theta_2(\theta_1+\theta_2), & -\theta_1^2\theta_2^2, & 0 \end{array}\right| \between d, 3c, 3b, a);$$

and, consequently, these same quantities, each multiplied by A^2, are given by the successive lines of

$$\left(\begin{array}{cccc} 0, & A^2, & -2AB, & -AC+4B^2 \\ -A^2, & 0, & AC, & -2BC \\ 2AB, & -AC, & 0, & C^2 \\ AC-4B^2, & 2BC, & -C^2, & 0 \end{array}\right| \between d, 3c, 3b, a).$$

Calling these X, Y, Z, W, that is, writing
$$X = 3A^2c - 6ABb + (-AC+4B^2)\,a, \ \&c.,$$
then X, Y, Z, W are the values of
$$\Theta_1-\Theta_2, \ \theta_2\Theta_1-\theta_1\Theta_2, \ \theta_2^2\Theta_1-\theta_1^2\Theta_2, \ \theta_2^3\Theta_1-\theta_1^3\Theta_2,$$
each multiplied by $A^2 \div (\theta_1-\theta_2)$; and the functions all four of them vanish if only $\Theta_1 = 0$, $\Theta_2 = 0$; or, what is the same thing, the equations $X=0$, $Y=0$, $Z=0$, $W=0$ constitute only a two-fold system.

The functions
$$\left(\begin{array}{ccc} X, & Y, & Z \\ Y, & Z, & W \end{array}\right)$$
contain each of them the factor $\Theta_1\Theta_2$, that is ∇; they, in fact, each of them vanish if $\Theta_1 = 0$, and they also vanish if $\Theta_2 = 0$; or, by a direct

substitution, we have

$$\begin{aligned} XZ - Y^2 &= \frac{A^4}{(\theta_1-\theta_2)^2} \cdot -(\theta_1-\theta_2)^2\,\Theta_1\Theta_2, && = -A^4\Theta_1\Theta_2, \\ XW-YZ &= \quad ,, \qquad -(\theta_1-\theta_2)^2\,(\theta_1+\theta_2)\,\Theta_1\Theta_2, && = -A^4\Theta_1\Theta_2\,(\theta_1+\theta_2), \\ YW-Z^2 &= \quad ,, \qquad -(\theta_1-\theta_2)^2\,\theta_1\theta_2\,\Theta_1\Theta_2, && = -A^4\Theta_1\Theta_2\,\theta_1\theta_2. \end{aligned}$$

Or, what is the same thing, these are $= -A\nabla$, $2B\nabla$, $-C\nabla$, respectively; thus the first equation is

$$\{3A^2c-6ABb+(-AC+4B^2)\,a\}\,\{2ABd-3ACc+C^2a\}$$
$$-(-A^2d+3ACb-2BCa)^2 = -A\,(A^3d^2+\&c.), = -A\nabla\,;$$

and similarly for the other two equations. The nodal curve is thus given by the twofold system $X=0$, $Y=0$, $Z=0$, $W=0$.

The method may be extended to the case where, instead of the quadric equation $(A, B, C \between \theta, 1)^2 = 0$, we have an equation of any higher order, but the formulæ are less simple.

Note on the Equation of the Two Planes which can be drawn through Two given Points to touch a Quadric. By H. M. TAYLOR, M.A.

[*Read June* 10*th*, 1880.]

If $U = ax^2+by^2+cz^2+d+2fyz+2gzx+2hxy+2lx+2my+2nz = 0$ be the equation of a quadric; and, in consequence,

$$W_1 = x\,(ax_1+hy_1+gz_1+l)+y\,(hx_1+by_1+fz_1+m)$$
$$+z\,(gx_1+fy_1+cz_1+n)+lx_1+my_1+nz_1+d = 0$$

the equation of the polar plane of the point $P_1\,(x_1y_1z_1)$; and if U_1 denote what U becomes when x_1, y_1, z_1 are written for x, y, z respectively; and $W_{1,2}$ denote what W_1 becomes when x_2, y_2, z_2 are written for x, y, z respectively; then it can be shewn that

$$\begin{vmatrix} U & W_1 & W_2 \\ W_1 & U_1 & W_{1,2} \\ W_2 & W_{1,2} & U_2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is the equation of the two planes which can be drawn through the two points $P_1\,(x_1y_1z_1)$, $P_2\,(x_2y_2z_2)$, to touch the quadric $U=0$; or, in other words, that

$$\begin{vmatrix} U_1 & W_{1,2} & W_{1,3} \\ W_{1,2} & U_2 & W_{2,3} \\ W_{1,3} & W_{2,3} & U_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is the condition that the three points $P_1\,(x_1y_1z_1)$, $P_2\,(x_2y_2z_2)$, $P_3\,(x_3y_3z_3)$ lie on a tangent plane of the quadric.

We may compare herewith the equation

$$\begin{vmatrix} U_1 & W_{1,2} \\ W_{1,2} & U_2 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which is the condition that the two points P_1, P_2 lie on a tangent line of the quadric.

If we write, in the last equation, $(x, y, z)\,(x_1, y_1, z_1)$ instead of $(x_1, y_1, z_1)\,(x_2, y_2, z_2)$, we have the well-known theorem that the enveloping cone drawn from the point P_1 to the quadric $U=0$ has for its equation

$$UU_1 - W_1^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If we draw an enveloping cone to this cone from the point P_2, it will degenerate into two planes, and will in reality be the two tangent planes to the quadric $U=0$ which pass through the two points P_1, P_2.

The equation of the enveloping cone from the point P_2 to the quadric $UU_1 - W_1^2 = 0$ will be

$$(UU_1 - W_1^2)\,(U_2U_1 - W_{1,2}^2) = (W_2U_1 - W_1W_{1,2})^2;$$

or, after simplification,

$$UU_1U_2 - UW_{1,2}^2 - U_1W_2^2 - U_2W_1^2 + 2W_1W_2W_{1,2} = 0 \ \ldots\ldots (5),$$

which may be written

$$\begin{vmatrix} U & W_1 & W_2 \\ W_1 & U_1 & W_{1,2} \\ W_2 & W_{1,2} & U_2 \end{vmatrix} = 0,$$

from which equation, by introducing a third suffix, we obtain the equation (2), which is the condition that a tangent plane to the quadric passes through the three points P_1, P_2, P_3.

Note by Professor Cayley.

The equation (3) is known as a transformation of the equation

$$\begin{vmatrix} & & x_1 & y_1 & z_1 & 1 \\ & & x_2 & y_2 & z_2 & 1 \\ x_1 & x_2 & A & H & G & L \\ y_1 & y_2 & H & B & F & M \\ z_1 & z_2 & G & F & C & N \\ 1 & 1 & L & M & N & D \end{vmatrix} = 0,$$

where $(A, B, \ldots)$ are the inverse coefficients of the system $(a, b, \ldots)$. And similarly (but which is in fact more obvious), the equation (2) is a transformation of

$$\begin{vmatrix} & & & x_1 & y_1 & z_1 & 1 \\ & & & x_2 & y_2 & z_2 & 1 \\ & & & x_3 & y_3 & z_3 & 1 \\ x_1 & x_2 & x_3 & A & H & G & L \\ y_1 & y_2 & y_3 & H & B & F & M \\ z_1 & z_2 & z_2 & G & F & C & N \\ 1 & 1 & 1 & L & M & N & D \end{vmatrix} = 0.$$

This last equation is, in fact,

$$(A, \ldots \between \alpha, \beta, \gamma, \delta)^2 = 0,$$

where α, β, γ, δ are the coefficients of x, y, z, 1 in

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_2 & 1 \end{vmatrix} = 0;$$

viz., $\alpha x + \beta y + \gamma z + \delta = 0$ is the equation of the plane through the three points; and the condition that this may be a tangent plane thus presents itself, in the first instance, in the foregoing form

$$(A, \ldots \between \alpha, \beta, \gamma, \delta)^2 = 0.$$

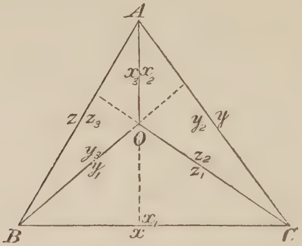
On the Focal Conics of a Bicircular Quartic.

By HARRY HART, M.A.

[*Read June 10th*, 1880.]

It is a well-known theorem, that a bicircular quartic may be generated in four different ways, as the envelope of a variable circle, whose centre moves along a given conic (the focal conic of the quartic), and cuts a given circle orthogonally. It has also been shewn that the common self-reciprocal triangle of each fixed circle, and its corresponding focal conic, has for its angular points the centres of the other three fixed circles; *i.e.*, if J, J_1, J_2, J_3 be the four fixed co-orthogonal circles, and F, F_1, F_2, F_3 the corresponding focal conics, the triangle formed by joining the centres of J_1, J_2, J_3 is the common self-reciprocal triangle of the focal conic F and the orthogonal circle J, &c.

Let O, A, B, C be the centres of J, J_1, J_2, J_3 respectively, O being that centre which lies within the acute-angled triangle formed by joining the other three, viz. A, B, C, so that J_1, J_2, J_3 are real and J is imaginary.



Let x, y, z be areal coordinates of a point, ABC $\left\{\begin{array}{l}\text{being the triangle of reference,}\end{array}\right.$

x_1, y_1, z_1 ,, ,, the same point, OBC ,, ,,

x_2, y_2, z_2 ,, ,, ,, ,, AOC ,, ,,

x_3, y_3, z_3 ,, ,, ,, ,, ABO ,, ,,

these different coordinates being connected by the equations

$$\left.\begin{array}{ll} x_1 \cot B \cot C = x & \\ y_1 \cot C & = z \cot C - x \cot A \\ z_1 \cot B & = y \cot B - x \cot A \end{array}\right\}, \text{ \&c.}$$

Then we may write the equations to the four fixed circles and their corresponding conics,

$$\left.\begin{array}{l} J \equiv \quad x^2 \cot A + y^2 \cot B + z^2 \cot C = 0 \\ J_1 \equiv -x_1^2 \cot A + y_1^2 \tan B + z_1^2 \tan C = 0 \\ J_2 \equiv \quad x_2^2 \tan A - y_2^2 \cot B + z_2^2 \tan C = 0 \\ J_3 \equiv \quad x_3^2 \tan A + y_3^2 \tan B - z_3^2 \cot C = 0 \end{array}\right\} \dots\dots\dots\dots (\mathrm{X}),$$

$$\left.\begin{array}{l} F \equiv p\, x^2 + q\, y^2 + r\, z^2 = 0 \\ F_1 \equiv p_1 x_1^2 + q_1 y_1^2 + r_1 z_1^2 = 0 \\ F_2 \equiv p_2 x_2^2 + q_2 y_2^2 + r_2 z_2^2 = 0 \\ F_3 \equiv p_3 x_3^2 + q_3 y_3^2 + r_3 z_3^2 = 0 \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots (\mathrm{Y}).$$

And if, for the point in question, (x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), we write s, s_1, s_2, s_3 to denote the squares of the tangential distances from the four circles respectively; and Δ, Δ_1, Δ_2, Δ_3 are the areas of the triangles of reference, it is easily proved that

$$\left.\begin{array}{l} s \;= 2\Delta\, J \\ s_1 = 2\Delta_1 J_1 = 2\Delta\, J_1 \cot B \cot C \\ s_2 = 2\Delta_2 J_2 = 2\Delta\, J_2 \cot C \cot A \\ s_3 = 2\Delta_3 J_3 = 2\Delta\, J_3 \cot A \cot B \end{array}\right\}.$$

Substituting for x_1, y_1, z_1 their values given above, we have

$$J_1 = -x_1^2 \cot A + y_1^2 \tan B + z_1^2 \tan C$$
$$= \{x^2 \cot A + y^2 \cot B + z^2 \cot C - 2x \cot A\,(x+y+z)\} \tan B \tan C,$$

or $$J_1 \cot B \cot C = J - 2x \cot A\,(x+y+z);$$

thus we have
$$\begin{aligned} s &= 2\Delta J, \\ s_1 &= 2\Delta\,\{J - 2x \cot A\,(x+y+z)\}, \\ s_2 &= 2\Delta\,\{J - 2y \cot B\,(x+y+z)\}, \\ s_3 &= 2\Delta\,\{J - 2z \cot B\,(x+y+z)\}; \end{aligned}$$

whence we obtain the identity

$$s_1^2 \cot B \cot C + s_2^2 \cot C \cot A + s_3^2 \cot A \cot B = s^2 \text{ .... (i.)}$$

Let us now first find the equation to the quartic, considering it as the envelope of a circle whose centre is on F and which cuts J orthogonally.

Let λ, μ, ν be the coordinates of its centre, then its equation is easily seen to be $J - 2\,(x+y+z)\,(\lambda x \cot A + \mu y \cot B + \nu z \cot C) = 0$,

which may be written $\lambda s_1 + \mu s_2 + \nu s_3 = 0$;

the envelope of this λ, μ, ν, being connected by the equations

$$\left.\begin{aligned} p\lambda^2 + q\mu^2 + r\nu^2 &= 0 \\ \lambda + \mu + \nu &= 1 \end{aligned}\right\},$$

is found in the usual manner to be

$$\frac{s_1^2}{p} + \frac{s_2^2}{q} + \frac{s_3^2}{r} = 0.$$

If we considered the quartic generated in a similar manner from J_1 and F_1, its equation would be $\dfrac{s^2}{p_1} + \dfrac{s_3^2}{q_1} + \dfrac{s_2^2}{r_1} = 0$;

or, in virtue of the identity (i.),

$$s_1^2 \frac{\cot B \cot C}{p_1} + s_2^2 \left\{\frac{\cot C \cot A}{p_1} + \frac{1}{r_1}\right\} + s_3^2 \left\{\frac{\cot A \cot B}{p_1} + \frac{1}{q_1}\right\} = 0.$$

Since this is the same equation as

$$\frac{s_1^2}{p} + \frac{s_2^2}{q} + \frac{s_3^2}{r} = 0,$$

we must have
$$\left.\begin{aligned} \frac{\cot B \cot C}{p_1} \qquad\quad &= \frac{1}{p} \\ \frac{\cot C \cot A}{p_1} + \frac{1}{r_1} &= \frac{1}{q} \\ \frac{\cot A \cot B}{p_1} + \frac{1}{q_1} &= \frac{1}{r} \end{aligned}\right\};$$

whence $$\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=\frac{1}{p}+\frac{1}{q}+\frac{1}{r},$$

and $$\left.\begin{aligned} p_1 &= p\cot B\cot C \\ q_1 &= -\frac{pr\cot C}{Q} \\ r_1 &= \frac{pq\cot B}{R} \end{aligned}\right\},$$

where

$$P=q\cot C-r\cot B,\quad Q=r\cot A-p\cot C,\quad R=p\cot B-q\cot A.$$

$$\left.\begin{array}{lll} \text{Thus } F_1 \text{ may be written} & & x_1^2-\dfrac{r}{Q\cot B}\,y_1^2+\dfrac{q}{R\cot C}\,z_1^2=0 \\ \text{similarly} & F_2\equiv & \dfrac{r}{P\cot A}\,x_2^2+y_2^2-\dfrac{p}{R\cot C}\,z_2^2=0 \\ \text{and} & F_3\equiv & -\dfrac{q}{P\cot A}\,x_3^2+\dfrac{p}{Q\cot B}\,y_3^2+z_3^2=0 \end{array}\right\}.$$

From the above definition of P, Q, R, it follows that

$$\left.\begin{array}{l} Pp+Qq+Rr=0 \\ \text{and} \quad P\cot A+Q\cot B+R\cot C=0 \end{array}\right\};$$

and, since $\cot A$, $\cot B$, $\cot C$ are essentially positive quantities, P, Q, R cannot have the same sign. Without loss of generality, we may assume P and Q of like sign and positive, and consequently R negative.

Then q is positive; for, if q be negative, p must be negative since R is so, and r must be negative since P is positive; thus p, q, r would be affected with the same sign, and F would be imaginary. Also, p is negative; for, if p were positive, r would have to be positive since Q is so, and again p, q, r would be of the same sign.

Thus P, Q, q *are positive, and* R, p *are negative.*

Let θ, θ_1, θ_2, θ_3 be the angles between the asymptotes of the four conics; then (Ferrers' "Trilinear Coordinates," 2nd Edit., Ex. 8, p. 98, altered for areal coordinates) we have

$$\tan^2\theta=-4\sin^2 A\sin^2 B\sin^2 C\,\frac{qr+rp+pq}{(p\sin^2 A+q\sin^2 B+r\sin^2 C)^2}$$

$$=\frac{-4(qr+rp+pq)}{4(qr+rp+pq)+P^2+Q^2+R^2-2QR-2RP-2PQ};$$

whence $$\sin^2\theta=\frac{-4(qr+rp+pq)}{M^2},$$

$$\cos^2\theta=\frac{M^2+4(qr+rp+pq)}{M^2}=\left\{\frac{p\sin^2 A+q\sin^2 B+r\sin^2 C}{M\sin A\sin B\sin C}\right\}^2,$$

where $$M^2 = P^2+Q^2+R^2-2QR-2RP-2PQ.$$

In the same manner, we find

$$\begin{aligned}\tan^2\theta_1 &= -4\sin^2 A\cos^2 B\cos^2 C\,\frac{q_1r_1+r_1p_1+p_1q_1}{(p_1\sin^2 A+q_1\cos^2 B+r_1\cos^2 C)^2}\\ &= \frac{4QR\,(qr+rp+pq)}{\{Q\,(p+q)-R\,(p+r)\}^2},\end{aligned}$$

and $$\sin^2\theta_1 = \frac{4QR}{p^2}\cdot\frac{(qr+rp+pq)}{M^2},$$

$$\cos^2\theta_1 = \left\{\frac{Q\,(p+q)-R\,(p+r)}{pM}\right\}^2,$$

and similar expressions for $\sin^2\theta_2$, &c.

Thus we have

$$\sin^2\theta_1 = -\frac{QR}{p^2}\sin^2\theta,\quad \sin^2\theta_2 = -\frac{RP}{q^2}\sin^2\theta,\quad \sin^2\theta_3 = -\frac{PQ}{r^2}\sin^2\theta;$$

and therefore, since R is negative and P and Q positive, when θ is real, θ_1 and θ_2 are so also, but θ_3 is imaginary; when θ is imaginary, θ_1, θ_2 are so also, and θ_3 is real; *i.e., of the four conics, three, viz.* F, F_1, F_2 *are ellipses, and the fourth* F_3 *is a hyperbola, or three are hyperbolas and the fourth an ellipse. In the case of a circular cubic, when the conics are parabolas, three of them,* F, F_1, F_2, *have their convexities in the same direction, and are cut in real points by the fourth* F_3, *which has its convexity in the opposite direction.*

Of the two supplementary angles between the asymptotes of each of the four conics, let us choose θ, θ_1, θ_2, θ_3 to be those which satisfy the equations

$$\cos\theta = \frac{p\sin^2 A+q\sin^2 B+r\sin^2 C}{M\sin A\sin B\sin C}$$

$$\cos\theta_1 = -\frac{Q\,(p+q)-R\,(p+r)}{pM},$$

$$\cos\theta_2 = -\frac{R\,(q+r)-P\,(q+p)}{qM},$$

$$\cos\theta_3 = -\frac{P\,(r+p)-Q\,(r+q)}{rM};$$

then, since it may be shewn that

$$[2\,(qr+rp+pq)\cot A-\{Q\,(p+q)-R\,(p+r)\}]\sin A\sin B\sin C = p\,(p\sin^2 A+q\sin^2 B+r\sin^2 C),$$

we have $$\left.\begin{aligned}2\,(qr+rp+pq)\cot A &= pM\,(\cos\theta-\cos\theta_1)\\ 2\,(qr+rp+pq)\cot B &= qM\,(\cos\theta-\cos\theta_2)\\ 2\,(qr+rp+pq)\cot C &= rM\,(\cos\theta-\cos\theta_3)\end{aligned}\right\}\text{.........(ii.)};$$

similarly

and

and hence also
$$\left.\begin{aligned} 2(qr+rp+pq)\,P &= qrM(\cos\theta_2-\cos\theta_3) \\ 2(qr+rp+pq)\,Q &= rpM(\cos\theta_3-\cos\theta_1) \\ 2(qr+rp+pq)\,R &= pqM(\cos\theta_1-\cos\theta_2) \end{aligned}\right\}.$$

In the limit, when the hyperbolas become parabolas, $qr+rp+pq=0$, and therefore $\theta, \theta_1, \theta_2, \theta_3 = 0$ together, or $= \pi$ together. Hence we conclude, by the principle of continuity, that $\frac{1}{2}\theta, \frac{1}{2}\theta_1, \frac{1}{2}\theta_2, \frac{1}{2}\theta_3$ are the angles made by the asymptotes with the same common axis (transverse or conjugate) of the conics; and, these being confocal, we may put

$$\cos\theta-\cos\theta_1 \equiv 2\left(\cos^2\frac{\theta}{2}-\cos^2\frac{\theta_1}{2}\right) \equiv -2\left(\sin^2\frac{\theta}{2}-\sin^2\frac{\theta_1}{2}\right)$$
$$= K(a^2-a_1^2),\ \&\text{c.},$$

where a, a_1, a_2, a_3 are the lengths of the semi-transverse axes of the conics.

Hence, writing x^2 in place of $x^2\cot A$, &c., so that (x, y, z), &c. are now no longer areal coordinates, equations (X, Y) become

$$\left.\begin{aligned} J &\equiv x^2+y^2+z^2=0, & F &\equiv \frac{x^2}{a^2-a_1^2}+\frac{y^2}{a^2-a_2^2}+\frac{z^2}{a^2-a_1^2}=0 \\ J_1 &\equiv x_1^2+y_1^2+z_1^2=0, & F_1 &\equiv \frac{x_1^2}{a_1^2-a^2}+\frac{y_1^2}{a_1^2-a_3^2}+\frac{z_1^2}{a_1^2-a_2^2}=0 \\ J_2 &\equiv x_2^2+y_2^2+z_2^2=0, & F_2 &\equiv \frac{x_2^2}{a_2^2-a_3^2}+\frac{y_2^2}{a_2^2-a^2}+\frac{z_2^2}{a_2^2-a_1^2}=0 \\ J_3 &\equiv x_3^2+y_3^2+z_3^2=0, & F_3 &\equiv \frac{x_3^2}{a_3^2-a_2^2}+\frac{y_3^2}{a_3^2-a_1^2}+\frac{z_3^2}{a_3^2-a^2}=0 \end{aligned}\right\};$$

and the equation to the quartic may be written

$$\frac{s_1^2(\cos\theta-\cos\theta_1)}{\cot A}+\frac{s_2^2(\cos\theta-\cos\theta_2)}{\cot B}+\frac{s_3^2(\cos\theta-\cos\theta_3)}{\cot C}=0,$$

or, in virtue of identity (i.),

$$\frac{s_1^2\cos\theta_1}{\cot A}+\frac{s_2^2\cos\theta_2}{\cot B}+\frac{s_3^2\cos\theta_3}{\cot B}=s^2\cos\theta.$$

In the case of a circular cubic, the equations to the four confocal parabolas become

$$\left\{\begin{aligned} &\text{since} && \frac{a_1}{a},\ \frac{a_2}{a},\ \frac{a_3}{a}\ \text{ultimately} = 1, \\ &\text{and} && a-a_1=\tfrac{1}{4}(l-l_1),\ \&\text{c.}, \\ & && l, l_1, l_2, l_3 \text{ being the latera recta} \end{aligned}\right\},$$

$$\left.\begin{aligned}
F &\equiv \frac{x^2}{l-l_1}+\frac{y^2}{l-l_2}+\frac{z^2}{l-l_3}=0\\
F_1 &\equiv \frac{x_1^2}{l_1-l}+\frac{y_1^2}{l_1-l_3}+\frac{z_1^2}{l_1-l_2}=0\\
F_2 &\equiv \frac{x_2^2}{l_2-l_3}+\frac{y_2^2}{l_2-l}+\frac{z_2^2}{l_2-l_1}=0\\
F_3 &\equiv \frac{x_3^2}{l_3-l_2}+\frac{y_3^2}{l_3-l_1}+\frac{z_3^2}{l_3-l}=0
\end{aligned}\right\}.$$

Let ξ, η, ζ be the coordinates of a point of intersection of J_3 and F_3, OAB being the triangle of reference. Then

$$\left.\begin{aligned}
\frac{\xi^2}{\cot A}+\frac{\eta^2}{\cot B}-\zeta^2\cot C=0\\
\text{and}\qquad -\frac{q}{P}\cdot\frac{\xi^2}{\cot A}+\frac{p}{Q}\cdot\frac{\eta^2}{\cot B}+\zeta^2=0
\end{aligned}\right\};$$

whence
$$\frac{\xi^2}{P\cot^2 A}=\frac{\eta^2}{Q\cot^2 B}=\frac{\zeta^2}{-R},$$

which shews that the points of intersection are real, since P, Q, and $-R$ are positive. In a similar manner it may be seen that the other conics do not intersect their corresponding circles in real points. *Thus, of the four conics, that which is different from the other three is the one that intersects its corresponding circle.*

Again, since p is $-$, and q is $+$, we see from equations (ii.) that $\cos\theta-\cos\theta_1$ and $\cos\theta-\cos\theta_2$ are of opposite signs, and that therefore $\cos\theta_1$, $\cos\theta$, $\cos\theta_2$ are in order of magnitude, *i.e., the conic F which corresponds to the imaginary circle J always lies between the two conics F_1 and F_2 which are of the same species.*

The coordinates of the centre of

$$F_1\equiv x_1^2-\frac{r}{Q\cot B}\,y_1^2+\frac{q}{R\cot C}\,z_1^2=0$$

are evidently
$$\mu_1,\quad -\frac{Q\cot B}{r}\,\mu_1,\quad \frac{R\cot C}{q}\,\mu_1,$$

where
$$\frac{1}{\mu_1}=1-\frac{Q\cot B}{r}+\frac{R\cot C}{q};$$

the result obtained by substituting these values in F_1 is

$$\begin{aligned}
f_1&=\mu_1^2\left\{1-\frac{Q\cot B}{r}+\frac{R\cot C}{q}\right\}\\
&=\frac{\mu_1^2}{qr}\,\{qr+rp+pq\}\cot B\cot C.
\end{aligned}$$

The coordinates of the centre of $J_1 = 0$ are $\tan B \tan C$, $-\cot A \tan C$, $-\cot A \tan B$; substituting these values in F_1, we have

$$\mathrm{f}_1 = \tan^2 B \tan^2 C - \frac{r}{Q} \cot^2 A \tan B \tan^2 C + \frac{q}{R} \cot^2 A \tan^2 B \tan C$$

$$= \left\{ 1 + \frac{Q-R}{QR}\, p \cot A \right\} \tan B \tan C.$$

Similarly $$f_2 = \frac{\mu_2^2}{rp} \{qr + rp + pq\} \cot C \cot A,$$

and $$\mathrm{f}_2 = \left\{ 1 + \frac{R-P}{RP}\, q \cot B \right\} \tan C \tan A,$$

since p is $-$ and q is $+$, f_1, f_2 are both positive, but f_1, f_2 are of opposite signs. Thus one circle is internal to (that is, on the same side of the conics as the foci), and the other external to, its corresponding conic; and, since these circles intersect in real points, it follows that *of the three conics, that one which is external to the other two is also external to, and encloses, its corresponding circle (which circle also might be shewn to be the one which encloses one of the ovals of the quartic); and that conic which is internal to the other two is external to, but does not enclose, its corresponding circle.*

F_1 is within or without F_2, according as r is negative or positive.

Let us now take for our triangle of reference that formed by joining any three of the points of intersection of J_3 and F_3. Then, ξ, η, ζ being areal coordinates, the equations to the circumscribing conic and circle

are respectively $$\left.\begin{aligned} F_3 &\equiv \frac{L}{\xi} + \frac{M}{\eta} + \frac{N}{\zeta} = 0 \\ J_3 &\equiv \frac{a^2}{\xi} + \frac{b^2}{\eta} + \frac{c^2}{\zeta} = 0 \end{aligned}\right\},$$

a, b, c being the sides of the triangle of reference. Also, let ρ, σ, τ be vectorial coordinates of any point, the angular points of the above triangle being the three fixed point circles; then the equation to a generating circle, the coordinates of its centre being ξ, η, ζ, is, since it cuts J_3 orthogonally, $$\xi\rho^2 + \eta\sigma^2 + \zeta\tau^2 = 0,$$

and the envelope of this, ξ, η, ζ being connected by the equations

$$\left.\begin{aligned} \frac{L}{\xi} + \frac{M}{\eta} + \frac{N}{\zeta} &= 0 \\ \xi + \eta + \zeta &= 1 \end{aligned}\right\},$$

is found in the usual manner to be

$$\pm \rho\sqrt{L} \pm \sigma\sqrt{M} \pm \tau\sqrt{N} = 0,$$

or $$\pm l\rho \pm m\sigma \pm n\tau = 0;$$

l, m, n being positive quantities.

This equation represents the four ovals (two real, two imaginary) of the quartic. ("Proceedings" of the London Mathematical Society, Vol. II., pp. 40, 41, where it is further shewn by Prof. Crofton that "if any one of the multipliers l, m, n exceeds the sum of the other two, one of the ovals is within the other. If each multiplier is less than the sum of the other two, the ovals are external to each other.")

That is, the ovals are external to each other or not, according as

$$-l+m+n,\ l-m+n,\ l+m-n,$$

are all positive, or one of them negative, *i.e.*, according as

$$(-l+m+n)\,(l-m+n)\,(l+m-n),$$

or

$$(l+m+n)\,(-l+m+n)\,(l-m+n)\,(l+m-n),$$

or

$$2m^2n^2+2n^2l^2+2l^2m^2-l^4-m^4-n^4,$$

is positive or negative.

But, replacing L, M, N by l^2, m^2, n^2 respectively, the equation to F_3 is

$$l^2\eta\zeta+m^2\zeta\xi+n^2\xi\eta=0,$$

and this is an ellipse or hyperbola according as

$$2m^2n^2+2n^2l^2+2l^2m^2-l^4-m^4-n^4$$

is positive or negative.

Hence we see that the ovals are external to each other or not, according as the conic which meets its corresponding circle is an ellipse or hyperbola.

On the Transformation of Elliptic Functions.[*]

By Prof. Felix Klein, of Leipzic.

In a series of papers which have appeared in the 14th and 15th volumes of the "Mathematische Annalen," I have investigated the transformations of the orders 5, 7, and 11 by a new method: finding in the first place, by geometrico-algebraical considerations, what are the magnitudes which in the several cases in question ought to be considered as the fundamental irrationalities. I will, as is most simple, express the results at once in the homogeneous form. There are, for $n=5$, 7, and 11, the ratios of 2, 3, and 5 magnitudes respectively, which are to be considered. For $n=5$ the two magnitudes η_1, η_2 are naturally independent*; for $n=7$ there is between the three magnitudes λ, μ, ν a single biquadratic relation

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[*] A Note offered by Dr. Klein to the Society, and accepted by the Council; the Secretaries are indebted to Prof. Cayley for the English translation.

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda = 0;$$

for $n=11$ there are between the five magnitudes y_1, y_4, y_5, y_9, y_3, in all 15 relations, which are (in this case also) biquadratic. They are obtained by equating to zero the several first minors of the Hessian determinant of

$$f = y_1^2 y_9 + y_4^2 y_3 + y_5^2 y_1 + y_9^2 y_4 + y_3^2 y_5.$$

I had as yet put all this only into an indirect relation with the ordinary theory of the doubly-periodic functions. But this relation is, as I lately found, an extremely simple one.† The ratios in question are nothing else than certain division-values of the Jacobian H-function

$$H\left(\frac{2Kx}{\pi}, q\right) = 2q^{\frac{1}{4}}\sin x \prod_{\nu=1}^{\nu=\infty}(1-q^{2\nu})(1-2q^{2\nu}\cos 2x+q^{4\nu}).$$

For, putting for any odd value of n

$$z_a = (-)^a q^{\frac{a^2}{n}} H(2aiK', q^n), \quad a = \{1, 2, \ldots (n-1)\},$$

then, in the first place, $z_{n-a} = -z_a$;

and there are thus only $\frac{1}{2}(n-1)$ values. These $\frac{1}{2}(n-1)$ magnitudes are precisely those which I have used for the cases $n=5, 7, 11$ respectively. And in fact, for $n=5$,

$$\frac{\eta_1}{\eta_2} = \frac{z_1}{z_2};$$

for $n=7$, $\lambda : \mu : \nu = z_1 : z_2 : z_4$;

and for $n=11$, $y_1 : y_4 : y_5 : y_9 : y_3 = z_1 : z_9 : z_4 : z_3 : z_5$.

Two things are, I hope, herein accomplished. First, my investigations for the cases 5, 7, 11 are rendered much more accessible than they were before. Next, the way is shown how to extend my investigations to the case of any larger value of n whatever.

ERLANGEN, *Oct.* 5, 1880.

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* The quotient $\frac{\eta_1}{\eta_2}$ is what I have called the Icosahedron-irrationality.

† I connected therewith considerations which are explained in a general manner in a Note presented the 3rd July, 1880, to the Munich Academy, and which is printed in the "Mathematische Annalen," t. xvii., pp. 133—138, and which afterwards M. Bianchi, in his paper in the same volume, pp. 234—262, has developed for the cases $n=3$ and 5. The expression obtained by M. Bianchi for the icosahedron-irrationality differs in form only from that given here in the text.

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APPENDIX.

In No. 556, *Nature* (Vol. 22, p. 170, June 24th, 1880), is an explanation by Mr. Frankland of Mr. Monro's difficulty in the light of Prof. Newcomb's Paper, "Elementary Theorems relating to the Geometry of a Space of Three Dimensions, and of Uniform Positive Curvature in the Fourth Dimension" (*cf. Proceedings*, Vol. viii., pp. 309, 310, and note). A rejoinder from Mr. Monro appeared in No. 558 of the same journal (p. 218, July 8th).

A Paper, on a subject allied to the above, by Ettore Ricordi, in the *Giornale di Matematiche* (Vol. xviii., pp. 255, 256), is entitled "I circoli nella Geometria non-Euclidea."

The expression for the surface of an ellipsoid first given by Prof. Tait, and communicated by Prof. Cayley to the Society (*Proceedings*, Vol. ix., p. 40), is obtained in a Paper by Mr. R. R. Webb, "Some Applications of a Theorem in Solid Geometry" (*Messenger of Mathematics*, Vol. ix., p. 173, 1880).

I have been requested by Dr. Zeuthen to include in the present Appendix the following remarks on his paper, "Déduction de différents Théorèmes Géométriques d'un seul Principe Algébrique":—

Le premier des "théorèmes géometriques déduits d'un seul principe algébrique," dans mon article à la p. 196 du Vol. x. des *Proceedings*, a été démontré antérieurement par M. Halphen à la p. 168 du Vol. ix. des *Proceedings*.

Le même savant géomètre, qui s'est occupé aussi, sans rien publier, du principe algébrique, et qui m'en a communiqué deux démonstrations algébriques, m'a fait observer aussi qu' à-côté de ses applications géométriques il en existe une application analytique intéressante. Euler a montré que dans le cas où une équation de second degré en x et en y, séparément,

$$Ax^2+2Bx+C = A'y^2+2B'y+C' = 0$$

a lieu, on a aussi l'équation suivante

$$\int_a^x \frac{dx}{\sqrt{B'^2-A'C'}} = -\int_b^y \frac{dy}{\sqrt{B^2-AC}}.$$

Le principe algébrique montre que l'une de ces intégrales elliptiques peut être déduite de l'autre par une substitution linéaire. Si, réciproquement, on déduit ce dernier Théorème de la Théorie des intégrales elliptiques, il fournirait une nouvelle démonstration du principe algébrique (de même que chacune des applications géométriques en fournit une). En tant cas le principe montre la *connexion* de ce théorème analytique et des différents applications géométriques.

"Cubic Determinants" (p. 17) are discussed in chapter vii. of Mr. Scott's "Treatise on the Theory of Determinants and their applications in Analysis and Geometry." This book has a very useful "List of Memoirs and Works relating to Determinants."

Prof. Cayley's paper on Prof. Clifford's "Theory of Distances" is printed in Prof. Clifford's *Mathematical Papers*, pp. 123—131.

Signor Brioschi writes, that some of the results obtained by Sir James Cockle (pp. 123—131), have been given in Memoirs by MM. Hermite, Füss, and Brioschi, but he has not forwarded the references.

I have received the following Note from the Author on the paper just referred to:—

In July last (1879), my brother, Dr. John Cockle, gave me Du Bourguet's work on the *Calculus* (Paris, 1810 and 1811, 2 vols., 4to). I then first became aware that Du Bourguet had (at pp. 75, 76 of vol. ii.) an article* containing a statement, illustrated by two examples, to the effect that $\square =$ a constant, say $\square = c$, may be a single solution, in x, y, z, of an unintegrable ternary. Multiplication into a variable factor, say β, will give the new ternary, which results from the multiplication, a new discriminoid, say $\square' \, (= \beta^2 \square)$. Now a given ternary may admit of a Du Bourguet solution $\square = c$, but, except in the particular cases $c = 0$ and $c = \infty$, the new ternary will not do so; for its solution is $\square' = c\beta^2 =$ a variable function. Hence, if the given ternary be that whose discriminoid is $\square'$, it would be vain, save in the excepted cases, to seek a Du Bourguet solution, and recourse must be had to other considerations (as to which see my paper "On Differential Equations," &c., *Proceedings* London Mathematical Society, Vol. x., pp. 105—120).

The Note which Mr. Merrifield read (pp. 45—47) simply gives some symmetrical cases of the orthogonal transformation, of a much more general character (but unsymmetrical) given by Prof. Cayley, and reproduced in Salmon's *Higher Algebra* (3rd edition, p. 39). The symmetrical form may be obtained from the one there given by writing

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* I have given a bibliography of Du Bourguet's work in "Notes and Queries" (5th Series, Vol. xii., pp. 182, 183), and a translation, of the article, in the "Manchester Proceedings" (Vol. xix., pp. 9, 10).

$$\frac{1}{\Delta} = k, \quad \frac{\lambda}{\Delta} = a, \quad \frac{\mu}{\Delta} = b, \quad \frac{\nu}{\Delta} = c,$$

and then putting $k = 0$. We thus get for the determinant the symmetrical form $\begin{vmatrix} a^2-b^2-c^2, & 2ab \quad, & 2ca \\ 2ab \quad, & -a^2+b^2-c^2, & 2bc \\ 2ca \quad, & 2bc \quad, & -a^2-b^2+c^2 \end{vmatrix}$,

the value of which is r^3, and the sum of the terms on the leading diagonal $= - r$, where $r = a^2+b^2+c^2$. The terms of this determinant will be integral, if a, b, c are either integral, or of the form *integer* $Z\sqrt{2}$, or indeed if they contain any common factor entering under the square root only.

It has been shown by Legendre and Gauss that every integer, or its double, is the sum of three squares. It follows that an orthogonal transformation of the above symmetrical character can be found for every whole number r. The transformation is, however, nugatory for certain low values of r.

The symmetrical transformation means a turn of two right angles about an axis whose direction cosines are proportional to a, b, c. That is to say, if a cube be taken, with the axes for edges, and these of rational length in a cubical system, it is always possible to find one or more axes, inclined to the coordinate axes, such that, if we turn the cube about them through two right angles, its points will still rest on points of the system.

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